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A Comparison of Mathematical Discourse in Online and Face-to-Face Environments

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A COMPARISON OF MATHEMATICAL DISCOURSE IN ONLINE

AND FACE-TO-FACE ENVIRONMENTS

by

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ABSTRACT

A COMPARISON OF MATHEMATICAL DISCOURSE IN FACE-TO-FACE AND ONLINE ENVIRONMENTS

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Master of Arts

Many studies have been done on the impact of online mathematics courses. Most studies concluded that there is no significant difference in student success between online and face-to-face courses. However, most studies compared "traditional" online and face-to-face courses. Mathematics educators are advocating a shift from traditional courses to student-centered courses where students argue and defend the mathematics under the guidance of the teacher. Now, the differences in online and face-to-face student-centered mathematical courses merit a more in-depth investigation. This study characterized student mathematical discourse in online and face-to-face Calculus lab sections based off of a framework derived from an NCTM standard for the students' role in discourse. Results showed that the discourse in both the face-to-face and online environments can be rich and productive. Thus, both environments can be viable arenas for effective mathematical discourse. However, this effectiveness is contingent on whether or not

the teacher as the facilitator can help the students avoid the ways in which online discourse can be impeded. The characteristics of discourse, how they compare, and the resulting recommendations for teachers are discussed.

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CHAPTER 1: RATIONALE

For many years, distance education has been an alternative to face-to-face education. An early form of distance education was correspondence course programs. Students would use the postal system to receive their assignments and return them after completion. The process would repeat for subsequent assignments until their educational goal was met. As new technologies have emerged, educators have adapted them for the benefit of distance education. For example, soon after television was invented, universities started to offer distance courses through it (Dutton, Dutton, & Perry, 2001).

Distance education served a minor audience until the development of the Internet. Distance education over the Internet is called online education. The public's use of online education has exploded over the past two decades and is projected to continue to do so in the future. Vasarhelyi and Graham (1997) found that in 1993 there were only 93 schools devoted to online learning, but in 1997 the number had jumped to 762. Engelbrecht and Harding (2005) estimated that the "e-learning market will grow from US$ 10.3 billion in 2002 to US$ 83.1 billion in 2006, and eventually swelling to over US$ 212 billion by 2011" (p. 235).

The growth of online education is due to its advantages over face-to-face education. Bartley and Golek (2004) stated that online education provides "unique alternatives for reaching larger audiences than ever before possible…traditional or non-traditional, full-time or part-time, and international—who perhaps have had limited access to advanced educational opportunities" (p. 167). They further stated that online classes, degrees, and certificates are especially valuable to those with demanding work, family, and social schedules. Big companies also use online courses as a means of training when they need to update the skills of their employees. In addition, Beard, Harper, and Riley (2004) mentioned that online education allows students to spend less time in class, to be able to continue their learning and complete their assignments at home, and spend less money on travel.

There are many advantages to online courses, but there are disadvantages as well. According to Coates, Humphreys, Kane, and Vachris (2004), younger undergraduate students who were found to lack the technological skills and discipline necessary to survive and participate in online courses felt inundated with course demands and soon dropped out. They also found that the average student dropout rate was generally higher for online courses. Piotrowski and Vodanovich (2000) also cited a few disadvantages. First, they mentioned that piracy issues were a disadvantage because when students submitted their work, there was no certain way to verify if they were the ones who did it. Second, they found that there were times when technological problems prevented access to the course materials. Third, they pointed out that these issues of piracy and course access often became more of a focus than course content.

Due to the growth, wide use, and perceived advantages and disadvantages of online education, the need to study this phenomenon has never been greater. We need to find out what the impact of online education has on the education of our students. Is it for good or ill? Or is it a combination of both?

The effectiveness of distance education has been studied by researchers in only one sense: student success (i.e. midterm test scores, final test scores, final grades, etc.). Research that compares students' learning in online and face-to-face environments has not strongly supported one side or the other. It has shown that there is no significant difference with regard to student success between online or face-to-face modes of delivering instruction (e.g., Akkoyunlu & Yilmaz Soylu, 2004; Aragon, Johnson, & Shaik, 2002; Brown, Stein, & Forman, 1996; Cooper, 2001; Ellis, Goodyear, Prosserz, & O'Hara, 2006; Russell, 1999; Smith, 2004).

If we look deeper into the setup of the majority of the online and face-to-face courses studied in the literature, we can see the reason why there is not much difference. This similarity stems from the fact that they are conducted in a traditional format. Traditional face-to-face courses typically feature a teacher lecturing to their students. Afterward, the students go home, do the homework, and learn the mathematics with their notes and book. This process is similar to the format of traditional correspondence courses which begins with instruction for the student through a video recorded lecture or other media. Then, after they receive the instruction, the students do the homework, and learn the mathematics with their notes and book.

Traditional online courses are essentially the same as traditional correspondence courses. Vasarhelyi and Graham (1997) sent out a questionnaire to 300 educators to find out what differences they perceived between correspondence and online courses. They found that almost 65% said that online courses were simply correspondence courses presented with new technology. Thus, the role of the student remains the same in any traditional delivery, regardless of the format. Therefore, there is little wonder that when researchers compared course grades, the differences between online and face-to-face learning were insignificant. The students were learning the material in the same way.

With the development of new learning theories, such as social constructivism (Palincsar, 1998), reform educators began to organize their classrooms and lessons in such a way as to allow their students to construct mathematical knowledge and ideas through social interaction (Ball, 1993; Lampert, 1990). As the students discussed new mathematical ideas they were either (1) able to situate the new knowledge or (2) encounter conflicts with their background knowledge or other students' interpretations of the new material. The teacher encouraged students to resolve these conflicts and as they did so, the depth of their knowledge of the mathematics increased (Perret-Clermont, 1980). The teacher intervened as he or she deemed necessary in order to assure that the student were headed in the right direction.

Students are able to situate new knowledge and resolve these conflicts through the process of discourse. As they discuss the new knowledge and the conflicts they see and reflect upon those discussions, they work out the differences through the explanations of their ideas and focusing on the mathematics (Cobb, Boufi, McClain, & Whitenack, 1997).

I attempted to find studies that specifically compared mathematical discourse in the face-to-face and online environments, but found nothing. Then, broadening my search terms, I looked for literature that compared just online and face-to-face discourse in any subject and discovered that the research was sparse. Therefore, I designed and implemented a study in which I characterized the discourse of students as they discussed mathematical tasks online and face-to-face. Based off of this characterization, I was able to compare the ways students discussed mathematics online and face-to-face and evaluate the online medium as an alternative to the face-to-face environment for discussing mathematics. With this approach, I was able to see that the environments are not as similar as the literature suggests.

CHAPTER 2: LITERATURE REVIEW

The purpose of this study was to characterize student mathematical discourse in online and face-to-face environments and thus compare how each facilitates discourse. The main body of literature relating to this investigation comprised studies that compared student performance in online and face-to-face classes through an analysis of their final grades or test scores. Thus, the guiding principle for this review of literature is to show that since the literature base only compared certain measures of student work (i.e., the *product* of student achievement) and not on the *processes* students used achieve their knowledge of mathematics in both media, their conclusions only showed a small picture of the comparison between face-to-face and online courses.

As an example of a weakness of comparing products of student achievement let us examine one popular measure of the product of learning: grades. Although grades can tell us a lot about how a student performed in the class, in the end they are not a consistent measure of knowledge. An "A" in the class, or course, would supposedly signify mastery of all topics. However, a "B" or a "C" would signify mastery of certain topics and not others, or an average mastery of all topics. Then, when compared with other sections of the same class, how does one know the exact reason for the lower grade? Grades also vary for the same class semester by semester. For example, other topics could be taught and the focus of the exams could also differ from class to class. If the class grades are curved, then they should not be compared owing to the differences between the class dynamics. Therefore, I believe that comparing grades or even test scores is not a deep enough analysis to truly to find the differences between the online and face-to-face environments. But, a comparison of an aspect of the students' processes of learning would.

As an illustration of the importance of investigating processes of student learning, or their learning experiences, we turn to Dewey (1916)'s definition of education. He stated that education was "the reconstruction or reorganization of experiences which add to the meaning of experience, and which increases ability to direct the course of subsequent experiences" (p. 76). Thus, Dewey (1916) advocated that the essence of education was in the experiences the students have and how they contribute and augment their overall life experience. He claimed that experiences with education would eventually culminate to an experienced person or citizen. Then, they would be capable of managing their own lives and have the ability to make sensible and informed decisions. One way to gain insight into the students' experiences of learning mathematics online and face-to-face is to study their discourse. The following review of pertinent articles will show the inconclusiveness of grades comparisons and how discourse would be a better criterion.

Online Versus Face-to-face Studies

The major body of literature relating to this project features studies that compared student achievement in the same class delivered in online and face-to-face environments. The studies compared various aspects of student performance like test scores, final grades, or a combination of them. In each study, the authors determined that there is no significant difference in student success when they compared test scores and final grades. Of course, there were a couple of exceptions found, but in the end, their impact was negligible.

No Significant Difference in Test Scores

The first type of research that compared face-to-face and online courses was one in which the authors made their conclusion of no significant difference based on a comparison of student test scores throughout and/or at the end of the course deliveries (e.g., Gagne & Shepherd, 2001; Gurbuz, Yildirim, & Ozden, 2001; Lim, Morris, & Kupritz, 2006; Rivera, McAlister, & Rice, 2002; Thirunarayanan & Perez-Prado, 2002). Rivera et al. (2002) investigated a traditional and a web-based section of an undergraduate course in management information systems[1]. This study can be considered representative of the studies that showed no significant difference in student performance based on test scores. The important comparison that the authors made was to compare an average of all the tests taken throughout the course. The authors used multiple-choice exams for all their sections, which consisted of questions pulled from a test bank. They felt that this test-making approach would allow the tests to be different enough to keep them secure, yet similar enough to be able to justify a comparison of the average scores. Table 1 illustrates how close the averages were.

Rivera et al. (2002) performed a *t*-test on the scores and confirmed their assumption that the difference was not statistically significant. The authors determined that these results boded well for the case that online courses were as good at face-to-face courses. Through all the problems and presumptions the authors had with the technology, they stated that it was surprising that the student performance in the web-based section did not falter.

[1] Rivera and Rice (2002) also investigated a hybrid version of their course, which was typical of a minority of the literature. However, only their work with the traditional and web-based sections is considered here.

Table 1

Exam Score Averages from Rivera et al. (2002)

Course Section	Exam Score Average
Traditional Course	74.85
Web Based Course	73.97

Note. From "A Comparison of Student Outcomes & Satisfaction Between Traditional & Web Based Course Offerings," by J. C. Rivera, M. K. McAlister, and M. L. Rice, 2002, *Online Journal of Distance Learning Administration, 5*(3).
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No Significant Difference in Final Grades

The second type of research that compares face-to-face and online courses is one in which the authors made their conclusion based on a comparison of the final grades that students earned at the end of the course (e.g., Dziuban, Hartman, & Moskal, 2004; Hodge-Hardin, 1997; Ryan, 1996). As a representative study in the area of the literature comparing final grades, Ryan (1996) compared traditional sections with distance education sections of Advanced Mathematics 3201 over the course of several years, 1992-1995. His main focus was to compare the final grades of all the students in each section. The combined number of students in the traditional classes far outnumbered the students taking the course through distance education. In order to compensate for this discrepancy, he randomly selected the same number of students as the distance course from the traditional course. Thus, his data were based on the performances of 38 students in 1991-92, 85 in 1992-93, 88 in 1993-94, and 104 in 1994-95. In order to compare the final grades, he did a 2 × 1 analysis of covariance, two groups and one factor. Table 2 gives his results. The adjusted scores reflected an average of the grades of the mathematics class in the study along with each student's English class and social studies class. However, the original scores did not differ significantly between the distance and traditional formats over the three school years.

Table 2

Analysis of Covariance Results from Ryan (1996)

	Group Means					
	Distance		Traditional			
Year	Observed	Adjusted	Observed	Adjusted	F-value	Sig.
1992-93	75.74	76.87	77.94	76.81	0.00	0.96
1993-94	76.92	78.79	79.03	77.17	1.27	0.26
1994-95	78.83	78.96	78.37	78.24	0.31	0.58
Combined	76.90	77.84	77.98	77.05	1.23	0.27

Note. From "The Effectiveness of Traditional vs. Audiographics Delivery in Senior High Advanced Mathematics Course," by W. F. Ryan, *Journal of Distance Education, 11*(2), p. 50.

No Significant Difference From Multiple Perspectives

The third area of the literature base compared student performance from more than one perspective (e.g., Dutton et al., 2001; Neuhauser, 2002; Tucker, 2001). These studies yielded more diverse results, but yet we could draw the same conclusion as the previous studies. Dutton et al. (2001) investigated an on-campus lecture section and an online section of CSC 114, Introduction to Programming in C++. The authors generated the data for their study using each student's final exam grade, course grades, and whether or not they finished the course. They found that the online average for the final exam was 73.5% versus the lecture class final exam average of 63.9%. Course grade averages also favored the online section, 79.0% versus 74.7%. However, the percent of those that completed the course favored the face-to-face group, 93.6% to 79.4%. This difference raises the question: Had the other online students been able to finish the course, what would have happened to the online data? I believe that it would have been at a similar level with the lecture section, because the more capable students most likely stuck through it. However, the authors decided to control for "lifelong learners" versus "undergraduates," which affected the significance. Doing so, they found that the difference ended up not being significant, thus confirming that school experience is a factor.

Neuhauser (2002) also conducted a course online and face-to-face. At the end of the courses, she noted that the average test scores for the online group was 88.1% and the face-to-face group was 86.2%. She performed a *t*-test and found that there was no significant difference. With respect to grades, she found that the online students achieved a 3.5 (on a 4.0 scale) and the face-to-face students, a 3.35. Again, there was no significant difference.

In addition to a comparison of test scores and grades, however, the author compared the learning modality preferences and Keirsy temperaments of all students in the course versus those who were successful (i.e., received an A or A–). First, she administered the Learning Modality Preference Inventory and found that 40% of the online students listed visual as their preferred style or one of their preferred styles. Also, 66% chose kinesthetic as their preferred style or one of their preferred styles. In the face-to-face environment, 43% of the students chose visual as their preferred or one of their preferred styles. Also, 43% of the students chose kinesthetic as their preferred or one of their preferred styles. With this data, her statistical tests indicated that there was no significant difference. I thought that it was surprising that the online section had more kinesthetic learners because one would think that a face-to-face class would allow opportunities for them to get hands-on experience. However, if the class were more traditional in nature, this would not be the case.

Second, the author administered a Keirsy temperaments inventory. She found that the online group was 59% sensation/judging, 3% sensation/perceiving, 24% intuition/feeling, and 14% intuition/thinking. The successful students in the group were 59% sensation/judging, 0% sensation/perceiving, 14% intuition/feeling, and 100% intuition/thinking. The author noted that it was surprising to see that all of the successful students in the online environment believed that they learned with the intuition/thinking temperament. With respect to the face-to-face group, 80% of the total group was sensation/judging, 6% sensation/perceiving, 7% intuition/feeling, and 7% intuition/thinking. We can see that most of the students that chose the face-to-face course were of the sensation/judging type. The successful face-to-face students were either sensation/judging or intuition/thinking.

This study shows that there are differences between the type of student that takes and is successful in an online course versus the type of student that sticks to the face-to-face option. This study also showed that a deeper investigation brought out some differences in the students based on the environment.

Tucker (2001) conducted a course with one section on-campus and the other through distance education. Her goal was to determine if distance education was as good as, better, or worse than traditional education. She performed statistical tests on a variety of types of data. Table 3 gives a portion of her results.

Table 3

Summary of Selected Results from Study by Tucker (2001)

	On-Campus		Distance Ed	
Variables	Mean	Std. Dev.	Mean	Std. Dev.
Pre-test	55.52	13.50	59.21	9.96
Post-test	65.55	10.91	72.43	9.12
Final Exam	78.26	12.63	85.92	8.16
Final Grade	80.57	16.16	85.42	13.11
Age	23.13	5.12	37.79	8.72
Homework	78.55	15.99	85.22	12.02
Research Paper	87.45	28.60	91.39	12.32

Note. From "Distance Education: Better, Worse, or as Good as Traditional Education?" by S. Tucker, 2001, *Online Journal of Distance Learning Administration, 4*(4).

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Tucker (2001) found significant differences for age, post-test scores, and final exam scores, all favoring the online group. She found that there were no significant differences in pre-test scores, research paper scores, homework scores, and the final course grades. However, she thought that the statistical insignificance of the pre-test scores was likely attributed to the classes having the same background knowledge of the subject in the beginning. I thought that the difference in age was the likeliest contributor to the numerical differences in the means of the homework scores, research paper score, and the final exam score. It was interesting that even though the age difference was great, there was little difference in the final course grades despite the fact that they did slightly better coursework. One might have thought otherwise. However, if the difference in the coursework scores were not statistically significant, then it is not likely that the difference in the final grades would have been either.

Differences in Online and Face-to-Face Discourse

My idea that there would be noteworthy differences in student activity online versus face-to-face through discourse stems from the literature. Although the literature strictly comparing online and face-to-face discourse in any subject was sparse, the few studies that I read presented similar ideas of how students felt about discussing class topics online and face-to-face.

Conversing Face-to-Face

The literature that compared online and face-to-face discourse did not discuss a lot about face-to-face conversations. Perhaps this lack of discussion is because we know much and we have much experience with conducting face-to-face mathematical conversations as a point of contrast. However, they stressed the powerful ability of the face-to-face environment to facilitate dynamic conversations. The students in Meyer (2003)'s study felt that conversations in the face-to-face environment were fast, had spark, energy, and enthusiasm. The students also had the ability to quickly build on one another's comments and to collaborate on the spot. This speed and energy the students had for discussion translated to high competition among the members of the group in the relatively short time in class. As a result, many wanted to discuss certain topics more and defend their ideas more thoroughly. Pérez-Prado and Thirunarayanan (2002) enjoyed that the face-to-face discussions more informal and interactive. They also found that the activities they did face-to-face were successful because they had the ability to time how long some took, an aspect pertinent to their survey. The ability to time a lesson was not possible online.

Conversing Online

The literature focused a lot on what students liked and disliked about holding online discussions. The major advantage to online discussions versus face-to-face ones was that the students could post any time. Each one of the studies found that the students and the teacher both enjoyed the ability to discuss their class topics at anytime, day or night.

One obvious characteristic of online conversation with a threaded discussion, discussion board, or listserv was that it is static. Because of this static quality, Meyer (2003) reported that her students thought online discussions were slow and boring compared with face-to-face conversations. Tiene (2000) said that half of his students felt that there was a loss of spontaneity discussing topics online versus face-to-face, and that this loss was not beneficial.

Another evident fact about conversing online is that the students must write instead of talk. Ellis (2001) found that her students would type everything out online as they would speak it face-to-face, thus forming the opinion that writing was as easy as talking. However, Meyer (2003) stressed that the ability to write decently was of extreme importance in her class's online discussions. With regard to the preference of writing versus speaking, Tiene (2000)'s study showed that this preference was split down the middle for his students.

One advantage to the online discussions was the ability to think about your response before you post it. Almost all of the students in Tiene (2000)'s study felt this ability was a positive attribute. As a result, Meyer (2003) found that her students saw that their conversations were more thoughtful, reasoned, and contained more evidence to back up their arguments. She also showed that there was less embarrassment possible in online discussions because students had time to think about the correctness of their ideas before they posted. In face-to-face discussions, students do not have that ability and sometimes they say something inaccurate causing them embarrassment. She also felt that the teacher could better answer their students' questions if given the time to think. In the face-to-face environment, the teacher would have to reflect on how best to answer the students' questions and deliver the answer all in a short period of time, which does not work well all the time. On the other hand, because the teacher has more time to do this process when working online, he or she would be able to give the most appropriate answer more often.

Another advantage that the online environment has, is the ability for the students to go back to any comment on the discussion board and read it. Ellis (2001) found that students felt that this ability was a positive attribute and were grateful that the details of the class discussions remained for them to review at any time. Tiene (2000) found that the majority of his students used this feature before they posted new comments.

One apparent drawback to the online environment was the inability to read the other group members' facial expressions, gestures, and conversational nuances as they discussed the class topic. Pérez-Prado and Thirunarayanan (2002) described them as *psychosocial cues*. The students in Ellis (2001)'s and Meyer (2003)'s studies attempted to use special characters and emoticons to capture some of these conversation actions they missed in their discussion. However, in the end, the students did not feel this attempt at a normal conversation worked very well. Tiene (2000) found that the majority of his students liked the ability to see gestures when they talked face-to-face, but half admitted that it was not a big disadvantage missing them in online conversation.

In conclusion, the studies that say there is no significant difference in student success in face-to-face and online classes, show that a focus on the students' end product do not give much insight into the differences in online and face-to-face courses. Nor would it aid us in assessing the educational value of the online medium for today's mathematics classes. From the literature, we only know that traditionally student success can be as good as the face-to-face environment with regard to student performance, but we do not know why or how. We also do not know how current teaching methods impact student mathematical discussions and consequently student learning. From the few studies done on face-to-face and online student discourse, there is evidence that if we examine the processes students go through to learn mathematics in both environments, such as discourse, perhaps we can reveal some important distinctions between discourse itself and learning in the two environments.

CHAPTER 3: THEORETICAL FRAMEWORK

This study focused on the characteristics of student discourse in two formats of delivering instruction, the first being an asynchronous online environment and the second a student-centered face-to-face environment. In order to study the characteristics of student mathematical discourse in these two environments, it is important to discuss the nature of online and face-to-face interactions, how students learn socially, and students' roles when conversing about mathematics. Looking at the students' role in discourse will give us an important framework for investigating the characteristics of student discourse, i.e., what students do or do not do as they discuss mathematics.

Nature of Interaction in Online and Face-to-Face Environments

A main theme from the literature review has been that there is no significant difference in student success in face-to-face and online learning. However, as we saw in the literature review, the courses investigated in those studies were "traditional" in nature. If we examine the nature of traditional online and face-to-face courses, we can see why there were no significant differences in student performance between them.

In the traditional face-to-face course, there is limited dialogue between the teacher and the students. The teacher dictates to the students the information they are to acquire and periodically poses questions to the class, each eliciting a single response, if correct. Throughout the class period, there are a number of students that have not answered questions or interacted much with the mathematics. The resulting discussion would not have permitted the students to assimilate the mathematics as well as a classroom whose atmosphere was perhaps a little different. A reform, or more student-centered, classroom has a specific goal to enable students to generate mathematical discussions. The students are given an opportunity to explore mathematical topics along with their peers under the guidance of the teacher. In this manner, they are able to have deeper discussions about the mathematics, and thus able to understand them better.

In traditional online courses, the students download their assignment, read how to do the problem set, work the problem set, and submit their answers back to the instructor. Questions are usually posed through email. If a student does use email, there is some discussion analogous to the face-to-face situation when the teacher asks the student a question. If the student does not contact the teacher through some means, there is no discourse occurring. One way to make this environment more student-centered is to give the students an opportunity to discuss mathematics through a discussion board. Then they would have a chance to have meaningful mathematical discussions with their classmates and thus fulfill their recommended role as outlined by Ball (1993), Lampert (1990), and NCTM (1991). Table 4 lays out the relationship discussed between online and face-to-face teacher-centered and student-centered approaches to teaching mathematics.

Table 4

Face-to-face Versus Online Course Descriptions in Light of Traditional and Reform Approaches

	Traditional (Teacher-Centered)	Reform (Student-Centered)
Face-to-Face	Lecture	Task based
	Little mathematical interaction	High interaction through group discourse
Online	Like correspondence course	Task based
	Little interaction	High interaction through discussion board

From the discussion above, we see that the students' level of discourse is low in the more traditional settings of both media. Overall, the way they learn in both environments is similar in part because of the level of discourse. It is for this reason, that the body of research has shown that there is no significant difference in student performance.

Mathematics educators today encourage a reform approach to the traditional, more teacher-centered classroom. They endorse a more student-centered classroom, one in which students can explain and prove mathematics to themselves and the teacher. Thus, the discourse of this type of classroom is fundamentally different than a traditional one. There are more student-to-student discussions, rather than teacher-to-student lectures. Because of this difference in traditional and reform approaches, the resulting differences between discussing mathematics in either media need to be characterized and investigated. As shown in the literature review, there is potential for differences to be found based on the very nature of discussing topics online and face-to-face.

Learning Socially

Students can reinforce their learning of mathematics through discussions of their ideas. These discussions would support the learning perspective of social constructivists and is a major component of the student-centered classroom, whether live or virtual. A follower of Piaget said, "Cognitive conflict created by social interaction is the locus at which the power driving intellectual development is generated" (Perret-Clermont, 1980, p. 12). This point of view means that as students work together, they will encounter differences in several areas: (1) their background knowledge with other student's background knowledge, (2) their background knowledge and the new material they are learning, and (3) their interpretation of the new material with other students' interpretations of the new material. In this study, the students will discuss these conflicts and their resolution will contribute to their intellectual development as Piaget thought. Thus, it is vital that the learning environment supports the aspects of discourse to allow this type of learning.

Students' Role in Discourse

The NCTM *Professional Teaching Standards* (NCTM, 1991) delineate the role that students are to fulfill in a reform classroom environment, whether face-to-face or online. Standard 3 is entitled, "Students' Role in Discourse." It states that,

> The teacher of mathematics should promote classroom discourse in which students—[1] listen to, respond to, and question the teacher and one another; [2] use a variety of tools to reason, make connections, solve problems, and communicate; [3] initiate problems and questions; [4] make conjectures and present solutions; [5] explore examples and counterexamples to investigate a conjecture; [6] try to convince themselves and one another of the validity of particular representations, solutions, conjectures, and answers; [7] rely on mathematical evidence and argument to determine validity (p. 45).

This particular view of the student's role in discourse captures the ideas from the literature about what students should be doing in a mathematics classroom with mathematical tasks.

These seven desired aspects or characteristics of student discourse are a vital part of this study in both environments. They can be consolidated to fit under three general aspects: (1) listening to, responding to, and questioning one another, (2) using a variety of tools, and (3) initiating problems and making and investigating conjectures and solutions. The first general aspect is comprised of when students listen to, respond to, and question one another (from first aspect of the NCTM's teaching standard on discourse) and initiating questions (from NCTM aspect 3). The second general aspect is comprised of using the teacher as a tool (from NCTM aspect 1), using a variety of tools to reason, make connections, solve problems, and communicate (NCTM aspect 2), using mathematical evidence and argument to determine validity (NCTM aspect 7). The third general aspect is comprised of initiating problems (from NCTM aspect 3), making conjectures and presenting solutions (NCTM aspect 4), exploring examples and counterexamples to investigate conjectures (NCTM aspect 5), and trying to convince themselves and one another of the validity of particular representations, solutions, conjectures and answers (NCTM aspect 6). The following is a description of how these general aspects are defined along with a description of how they can be evidenced in the online and face-to-face environments.

Listening To, Responding To, and Questioning One Another

Students participate in one of three roles when they discuss any topic. They are: (1) listening, (2) responding, and (3) asking questions. When students are truly listening to someone speak, they record in their minds the important points that the speaker makes, in order to comment on them when it is their turn to speak. The subsequent comments made by the listener are called the response. The response can expound, clarify, agree, or disagree upon the important points that the listener remembered. One special type of response that can be made by a student is a question. Questions are used in conversation in order for the listener to clarify the responses given in the conversation or to pose new ideas. These clarifications help the listener to better understand the speaker's important points and help the speaker think about different ideas.

In the face-to-face environment, students are able to listen to, respond to, and question one another in a dynamic manner. Listening in this environment can involve full or partial focused attention between the speaker and the listener. Listening goes hand in hand with responding. Typical responses from the listener include gestures like nodding. They can also give responses like "Uh-huh," or "Yeah," to indicate that they are paying attention. When the speaker is finished, the listener immediately has the opportunity to comment on his or her important points. Also, questions are usually answered without delay. In this manner, a dynamic discussion is started and continued.

In the online environment, students are able to listen to, respond to, and question one another in a static manner. A longer period of time elapses between the students' exchanges in conversation. Listening has a different quality than it does face-to-face because the conversation is online. When a student reads another's posts, they are essentially listening to what they had to say as if their comment was verbalized. Any evidence of listening would be in the subsequent responses within a discussion thread. If the students were paying attention to the discussion, they would make references to the points previously brought up and no posts would repeat the same answers. They could handle questions similarly in this environment, because they could be asked at any point in the conversation. Then students can post their answer directly under it in the thread, which would probably not disrupt the flow of conversation. However, the turnaround time for questions could be hours or even days.

Using a Variety of Tools

It is advantageous and recommended that students use a variety of tools in order to facilitate their computations of and conversations about mathematics at any level. NCTM specifically wrote a standard on using a variety of tools in student discourse. It is Standard 4 of the NCTM *Professional Teaching Standards* (NCTM, 1991) entitled, "Tools for Enhancing Discourse." It states that,

> The teacher of mathematics, in order to enhance discourse, should encourage and accept the use of—[1] computers, calculators, and other technology; [2] concrete materials used as models; [3] pictures, diagrams, tables, and graphs; [4] invented and conventional terms and symbols; [5] metaphors, analogies, and stories; [6] written hypotheses, explanations, and arguments; [7] oral presentations and dramatizations (p. 52).

There are many examples of each tool mentioned in the article. There also many ways each one of these tools can be used. I briefly describe a few examples of tools and how they can be used in mathematical discussions.

Computers can run programs that are great at calculating and displaying data and functions too advanced for the calculator. It can be programmed as a tutor, for games, or for microworlds. The calculator is essentially a miniature computer. Doerr and Zangor (2000) did a study on students' use of calculators as a tool. They found that there were five roles it played in the classroom: (1) a computational tool to compute answers, (2) a transformational tool to change the difficult procedures to easier ones, (3) a data collection and analysis tool to gather data and investigate it, (4) a visualizing tool to see representations of functions, etc., and (5) a checking tool to verify the work of others.

Other technology may include calculator based laboratories, overhead projectors, or video presentations.

Concrete materials are used as representations of abstract mathematical ideas in order to gain understanding about them. Some examples of concrete materials are base-10 blocks, cones that come apart to show conic sections, algebra tiles, dice, coins, and constructed polyhedra, to name a few. As students discuss abstract concepts, they can use these concrete materials to explain their thinking or model a problem to show their solution to it.

Pictures, diagrams, tables, and graphs are used in discussions to organize and display information in order to assist in collecting and explaining data. Using a picture or a diagram of a parabola greatly assists the corresponding algebraic representation of it. A table of numbers can help show the variation as well. Students discuss these visuals in the same way they discuss concrete materials by referring to them often and making and explaining conjectures about their meaning.

Invented and conventional terms and symbols are very important to discussing mathematics. They are the principal medium with which to communicate mathematical ideas on paper or online. For example, when students discuss integral in writing, they use a $\int$ before the function and a dx after it, if the function is in terms of x. This description shows the conventional way to express an integral. If a student did not know the conventional way, they might write, "Int($f(x)$)," which could be an invented way the student uses for integral of a function. Either way is fine, but if the student wishes to communicate in writing the integral of something to someone, then he or she would have to use the conventional way, or the way everyone else does for that person to understand.

If not, the other person reading Int($f(x)$) might interpret it as some integer part of the function.

Metaphors, analogies, and stories are good tools to use to explain a concept or a student's understanding of the concept. Metaphors and analogies make comparisons between objects or ideas. For example one can make comparisons between fractions and rational functions in order to explain operations with rational functions. Students can use stories to explain how they learned a certain mathematical topic or the history behind it. This type of explanation could allow another student to gain understanding the same way as the one who told the story.

Written hypotheses, explanations, and arguments are great for recording an idea generated by students so they can use it at a later date. The recorded hypothesis, explanation, or argument must be used in mathematical discourse. And by writing them down, students can practice invented and/or conventional terms and symbols. Explanations and arguments are especially used to establish the validity of a student's mathematical notion. This idea relates back to Dewey (1916)'s view of education. Through explanations and arguments, students resolve their conflicts with their background knowledge, their interpretation of new material and their peers' interpretation of new material. Mathematical journals are also good ways to compile these hypotheses, explanations and arguments.

Oral presentations and dramatizations are good tools for discussions and good for generating discussions, too. Oral presentations at the board, for example, can be used in class discussions where students can debate about several ideas presented as solutions to a given problem. Dramatizations are presentations that are given in vivid or striking ways. They are good to give variety to the normal classroom activities and can be effective in acquiring information because it is encoded with the special way it was presented.

In the face-to-face environment, all these tools are available for use and the environment allows them to be used easily. This is the environment that serves as the basis from whence these tools were derived. They have the ability to discuss and use technology, models, diagrams, invented and conventional ways to write, and explanations with each other in the same space and time.

On the other hand, the online environment does not facilitate such uses in all areas. For example, students use computers and technology to discuss their ideas, but it is not clear how much a calculator or concrete materials could be explicitly used. Evidence of the use of these tools could only be inferred from the posts since we cannot see if the student used them while working out problems. Diagrams could be posted and discussed, but such discussions would not be synchronous, and thus their cohesiveness is unknown. In a discussion board, conventional terms and symbols can be used through text and mathematical equation editors. If one writes mathematical arguments in text, they are likely to use a lot of invented terms and symbols because text is limited in its mathematical fidelity. Explanations and arguments can also be used and will always be available for the students to read at any time. Students must use these tools in their posts. The goal for discussing mathematics as students is to use explanations and arguments to convince one another of mathematical conjectures. Oral presentations and dramatizations would not be possible with a discussion board following the true sense of the words. However, the environment does allow for students to use written versions. Creative students would be able to convert a dramatization to words and post it. But, overall, writing out a presentation or dramatization takes time. It is not clear if online students will take the time to do it.

Initiating Problems, Making and Investigating Conjectures and Solutions

This sequence of actions models the usual approach taken by students to solve mathematical tasks. First, they initiate problems, which can be done in a couple of ways. A student may ask for initial thoughts on how he or she or the group can approach the problem. They could also work on the problem individually, and then talk with each other about their initial solutions. What is important at this stage is to find out how they begin discussing their tasks. After they initiate their problems, they have debates or give explanations of how the task should be solved. This stage is called making and investigating conjectures. Eventually the students would agree on a solution. This solution can also go through a rigorous investigation. This stage is called making and investigating solutions.

In the face-to-face environment, students have the opportunity to initiate problems by negotiating the direction they will take in a mathematical task quickly because they are together in the same room at the same time. They can also converse about their initial ideas for the solution of the problem by showing each other their work and working out their differences together. They can make, explain, and negotiate conjectures so all can understand and agree. Eventually, a common idea will surface, which the group can take as their solution. The students do this one problem at a time in one sitting until the task is complete.

In the online environment, all the above things can be done, but it all takes more time to develop. The students are not online all the time. Therefore, they will be intermittently attending to the same process the face-to-face group takes in order to complete the lab. As they initiate their task they must consider this lag time in order for them to finish on time. They must also make an initial choice of how much of the lab they want to complete in the first post to the group. They must achieve a balance of covering enough information, but not too much, in order to have a cohesive discussion. If the right amount of material is proposed for discussion, then the next stage of making and investigating conjectures can go well. Students can formulate explanations and arguments to prove their conjectures as they type out their post before they are entered into the conversation. This ability is something that is not really possible in the face-to-face environment. In this way, the online students can make and agree upon correct solutions in their discussions.

Research Question

What are the characteristics of student mathematical discourse in the online and face-to-face environment and how do they compare?

CHAPTER 4: METHODOLOGY

The purpose of this study was to compare student mathematical discourse through two different modes of delivery, that of online and face-to-face. I studied certain aspects of their conversations, outlined in the theoretical framework, as they discussed the topic of derivatives in a calculus course. What I studied and how I studied it is the subject of this chapter.

Course Setup

I investigated student conversations about derivatives in an Introduction to Calculus (Math 119) course at a large university located in the Western United States. The course was set up specifically for students who majored in agriculture, biology, business, economics, management, among others. It covered the introductory topics of the traditional three-semester calculus sequence in one semester, with applications from current issues in life sciences and business. The course's textbook was *Calculus with Applications, 8th ed.* (Lial, Greenwell, & Ritchey, 2005).

The class had approximately 180 students, met five days a week, and the students received four credits. On Mondays, Wednesdays, and Fridays for one hour, the students met for a lecture provided by an instructor in a lecture hall. The lecture portion was used as a time to introduce and explain the topics of the course. The instructor did so with a practical application and then studied further examples, illustrating what rules governed the concept being taught. In an effort to give as many students as rich of an experience as possible, the instructor selected six to ten different students to sit in the front row each class period. During those classes, the instructor learned the names of the students, specifically asked them questions, and taught the lesson as if they were the class. The students on the other rows were able to participate as well.

The class was also divided into six lab sections of 20 to 30 students each. Those lab sections met every Tuesday and Thursday in separate classrooms to do related assignments, which gave the students an opportunity to practice the material introduced in the class lecture. For this study, I conducted two labs sections. One lab section with 23 students did the assignments in a face-to-face environment. The other lab section with 26 students did the assignments online.

The Face-to-Face Section

On Tuesdays, for about an hour, I met with the face-to-face lab section in a classroom where I demonstrated some homework problems, administered a quiz, and collected the homework from the lecture. I conducted each lab by running a demonstration of student-selected homework problems, carried out either by the students themselves or myself. This activity constituted the majority of the lab class. During the last 20 minutes of the period I passed out quizzes. Quizzes consisted of three problems similar to the homework exercises. At the end of the period, the students submitted their quiz and homework, consisting of ten to 15 drill problems and four application problems, per section of the textbook. Approximately two sections of homework were due each week.

On Thursdays, for about an hour, the face-to-face lab section met in the same classroom and did task-based assignments, called labs, which were related to the current lecture topic. The classroom desks were set up in groups of five so the students could work together on their labs. The students completed the labs in groups of their own choosing, except for the focus group. One set of five desks was designated to be the area for a focus group to be video recorded. The students that participated in this group did so of their own volition and knew that they were being taped. It was my intention to maintain the same five students in the focus group; however, if one of the students was absent, showed up late, or declined to participate, I would ask another student to fill in. The original five students were Josh, Jayden, Kim, Kevin, and Gary[2]. They worked together on Labs 1 and 2. For Lab 3, Austin replaced Kim and Calvin replaced Kevin. For Lab 4, the original five returned except Ashley replaced Gary.

The Online Section

I conducted the second lab in an online format with all activities being done online. We used a computer program called the Blackboard Learning System (2008)[3] to facilitate all our online needs. I trained the online students in discussing mathematics in this environment by setting aside some time during the first few Tuesday and Thursday lab days.

First, I showed them Blackboard's Discussion Board feature. I set up this part of the program with many little discussion boards as shown in Figure 1. The first discussion board link, entitled "Homework," allowed the students to ask the teacher for help on any homework problem, creating one thread for each. The next discussion board link was "Topic Discussions" for discussions of any content topic like derivatives, or the product rule. I set up a third discussion board called "Labs" for general discussions of lab topics like the limiting process table from Lab 2 (see Appendix A). Finally, I established specific discussion board links for each of the groups for each of the labs.

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[2] Names used in this thesis are pseudonyms.
[3] The lab was conducted using Blackboard 7.3. After this study the university upgraded to Blackboard 8.0. All screenshots were taken from the new version of Blackboard.

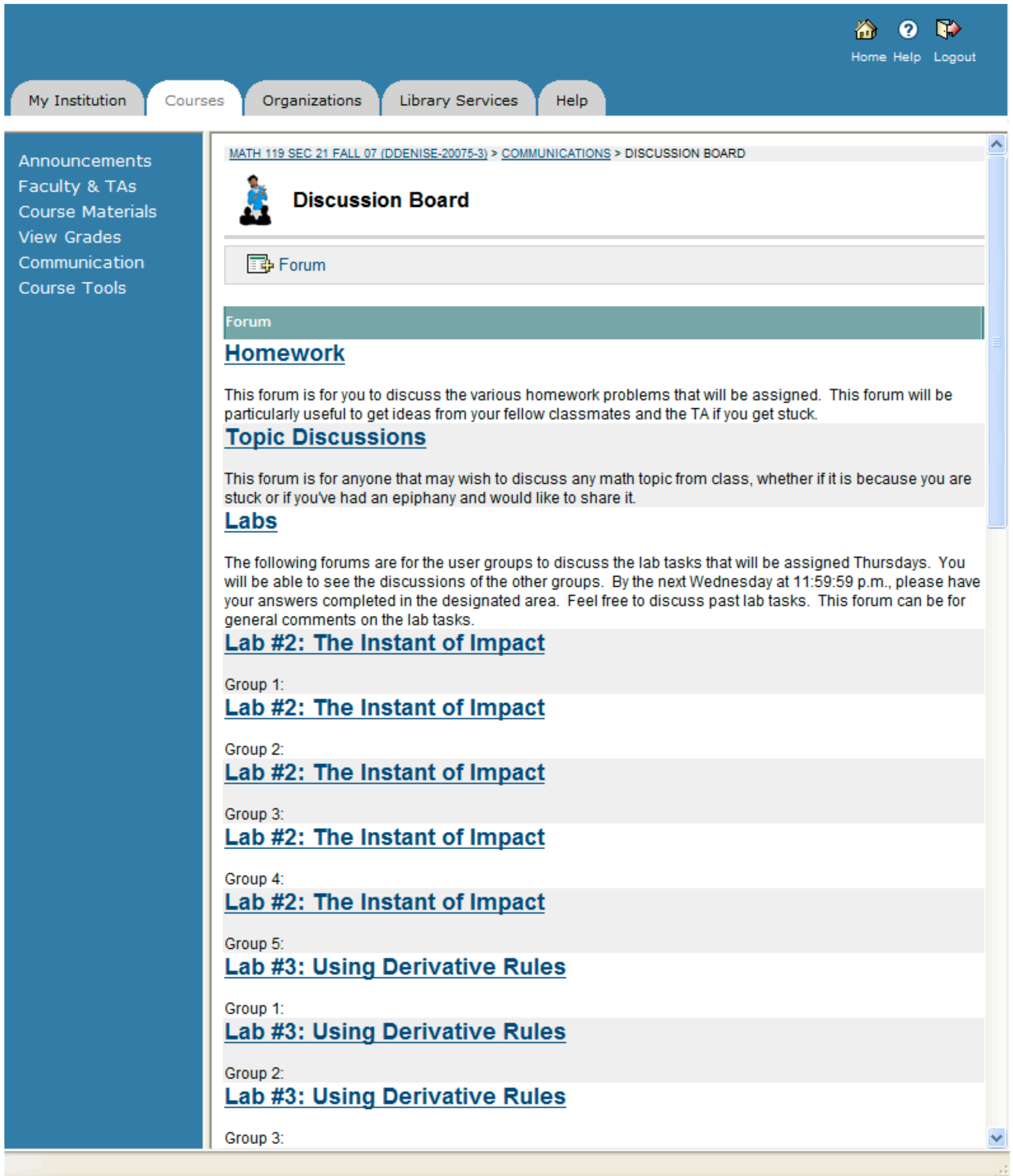


Figure 1. Top level discussion board screen.

Inside the discussion boards for the groups, there were three threads set up (see Figure 2). The first thread listed the lab questions for the students to answer. The second thread was where their final answers were to be posted. The third thread was for group discussion. An example of a threaded discussion is seen in Figure 3.

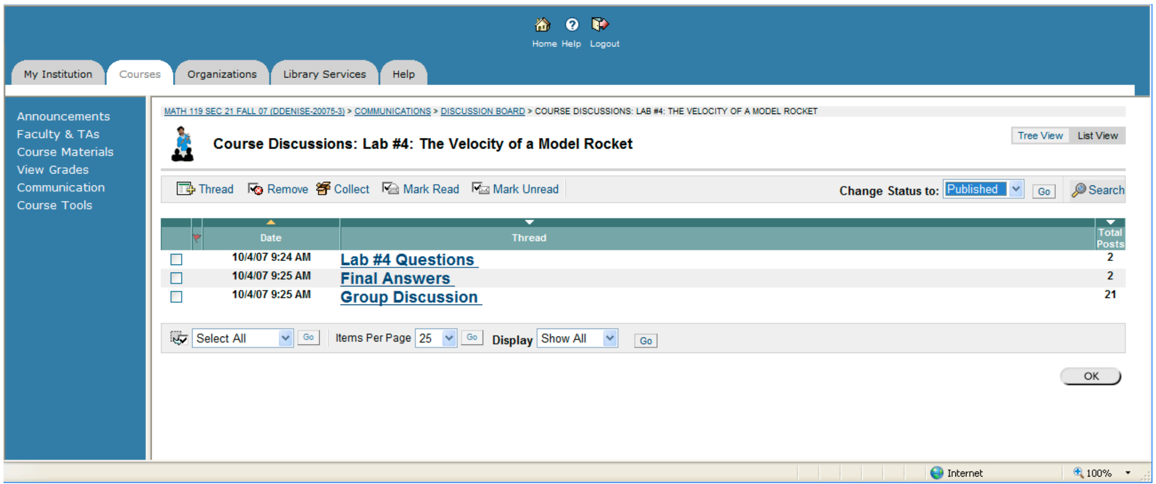


Figure 2. Individual lab discussion board.

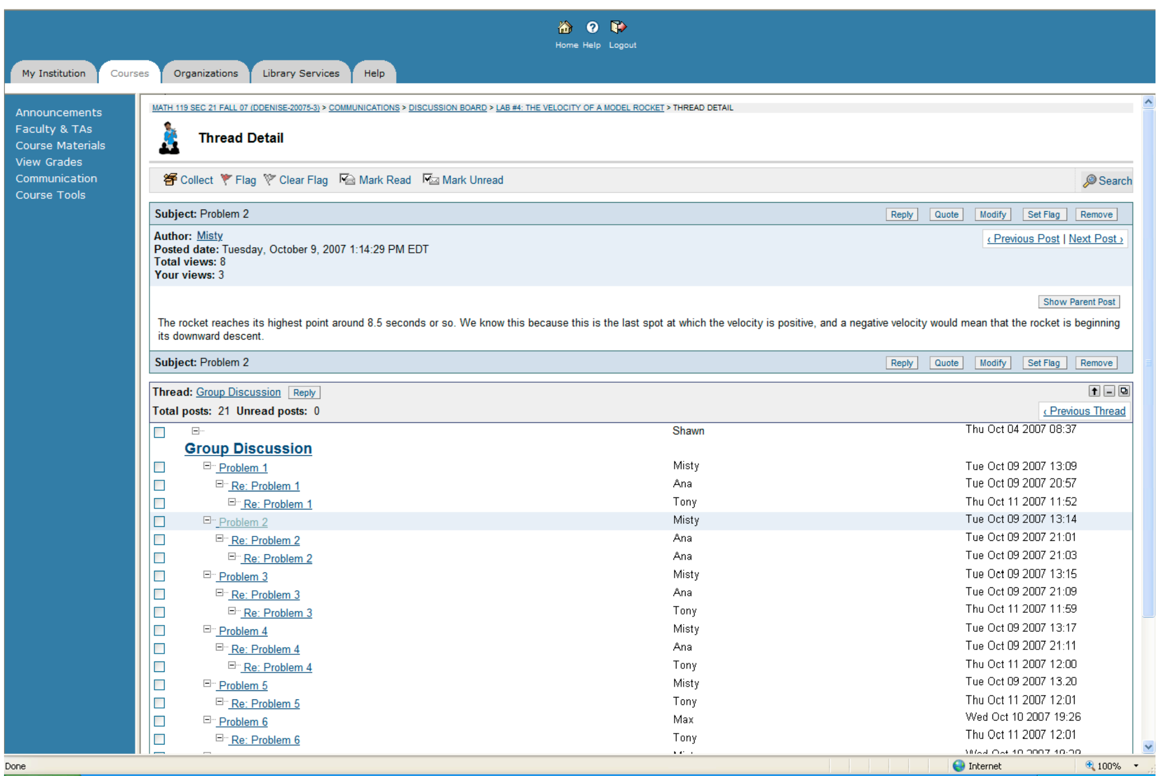


Figure 3. Group discussion threads.

As in the face-to-face lab section, I scheduled Tuesdays to demonstrate some homework problems, administer a quiz, and collect homework. The students and I led demonstrations of any homework problem through the "Homework" discussion board, mentioned previously. While the other lab section was only able to discuss homework on Tuesdays, the students and I eventually posted questions and solutions at our convenience. Because we worked online, we did not have to wait until the next Tuesday to discuss homework questions. Typical homework posts asked questions like, "How do you do Question 15 from Section 3.4?" or "In all my problems from the homework, I've been having issues with…"

Quiz length and questions were identical to those I gave in the face-to-face section. I handled all of the creation, administration, and grading of the quizzes through Blackboard's Test Manager. The homework length and questions were also identical to the other lab section. However, to turn in the homework the students created an electronic version of their solutions and uploaded the file to Blackboard's Digital Dropbox, where I was the only one able to view and grade them.

On Thursdays, I posted the same lab tasks as the face-to-face section to the lab questions link on the lab discussion board, as shown in Figure 2 (p. 34). In the beginning of the study, I had divided the section into four groups of five and one group of six. They remained in those assigned groups for all the labs. The students in the group knew they were discussing the labs for a research project and participated on their own volition.

They discussed the lab tasks as a group using the group discussion thread of the lab discussion board (see Figure 3, p. 34). This discussion took place from the Thursday the questions were posted, through to the next Wednesday. In order for me to grade the students on their participation, I established a grading deadline. The students could post ideas anytime, day or night. Then, by Wednesday at 11:55 p.m., a "spokesperson," selected each week, was to encapsulate the final answers of the group and post them on the discussion board under the final answers thread. Usually by the next morning, I went through all posts and graded their answers and level of participation. Even though the students posted their final answers and I had graded their discussion, I invited the students to continue their conversations through to the end of the semester. Some students followed that invitation (see post dates of comments in Figure 3, p. 34).

Course Content

The course began with a review of functions for the first week-and-a-half. Then they studied derivatives. This was the content topic which was the subject of the conversations that I studied in this project. Table 5 lists the specific chapters covered, their topics, and how the Thursday labs were situated within the content. After derivatives, they studied integrals and various selected topics from the second and third semester calculus courses to finish out the course.

Table 5

Relative Position of Labs With Respect to the Course Content

Week	Chapter	Topics	Lab[a]
1	1	Linear Functions	
	2	Nonlinear Functions	
2	3	The Derivative	Lab #1: Families of Functions
3 & 4	4	Calculating the Derivative	Lab #2: The Instant of Impact
			Lab #3: Using Derivative Rules
5	5	Graphs and the Derivative	Lab #4: The Velocity of a Model Rocket

[a]Lab questions for Labs 2, 3, and 4 are found in Appendixes A, B, and C.

Data Collection

The data for this study consisted of students' posts on a discussion board and video recordings of student discussions about mathematics respectively in the online and face-to-face environments. I made an initial plan for collecting the data and decided to implement that plan with the students while they worked on Lab 1 in both environments as a trial run. With regard to the face-to-face students, I set their desks into circles of five each. Then, for the focus group, I placed a table microphone on one of the desks and plugged the other end into the video camera and recorded their discussion of the first lab. The trial run went well, except for the audio level from the video recording of the face-to-face group was low. It was hard to hear their conversation. Therefore, in the subsequent labs, I specifically asked the students to speak louder while working. With regard to the online students, I set up the various discussion threads as shown in Figures 1-3 (pp. 33-34). Each group had five students and was instructed to post their comments in their assigned thread, but, if desired, they were able to go and read or post in other conversations. This approach worked well. The only adjustment that I needed to make was to reassign the groups because some students had dropped the course after the first week.

After the trial run with Lab 1 in the face-to-face section, I had Labs 2, 3, and 4 video recorded in the same manner. I had the audio transcribed along with descriptions of the students' actions that were captured visually. These transcriptions were then transferred to a spreadsheet for coding. Similarly, after the trial run with Lab 1 in the online environment, I transferred the text from the discussion board from the work of all five groups in Labs 2, 3, and 4 to a spreadsheet. Therefore, the dialogue from the discussions of Labs 2, 3, and 4 from the face-to-face and online students, comprised the data set for the study.

Data Analysis

After the students finished learning about derivatives in the lecture and labs, I began analysis of the data set. I coded the conversations along the three aspects of discourse as given in the theoretical framework, which were derived from NCTM (1991). They were: (1) listening to, responding to, and questioning one other, (2) using a variety of tools, and (3) initiating problems and making and investigating conjectures and solutions. I read though the data multiple times for each aspect in order to identify the parts of the data that illustrated the aspects of discourse as described in the literature. The following discussion describes how I accomplished this analysis.

Listening To, Responding To, and Questioning One Another

It took several revisits to the data to identify what types of examples I should look for with regard to this aspect. It was evident in both environments that students listened and responded to each other. However, the ways in which they asked each other questions were different. For this reason, I decided to break this aspect up into two parts and review the data looking at each separately: (1) listening and responding to one another and (2) questioning one another.

Listening and Responding to One Another

On the surface, it was apparent that the students were able to have some good discussions of the mathematical tasks in either environment. Upon further investigation, I realized that there were occasions when the online students did not listen (e.g., students failing to answer questions or students repeating posts) and did not respond (e.g., students procrastinating their work until the end). Then I reviewed the face-to-face data and also saw that there were occasions when students did not listen (e.g., students again failing to answer questions) and did not respond to one another (e.g., group not picking up and discussing certain ideas proposed by students). Therefore, my initial impression was that either medium had noteworthy impediments for students to listen and respond to each other effectively. Afterward, I reviewed the data objectively and counted the number of times a question was not answered in either environment, when online students repeated themselves, when online students procrastinated, and when face-to-face students ignored ideas. These events actually occurred a relatively small number of times. Therefore, no further or deeper analysis was needed.

Questioning One Another

Next, I reviewed the online data and specifically looked at each question that was asked. After reading over the data several times, it was soon evident that there were only a few types of questions. I thought about each question's content and function. I noticed that some were mathematical in nature and some were not. I started with this distinction. I thought about the mathematical questions and noticed that some were very specific on the mathematics they questioned. Others were very general and simply asked whether or not their mathematics was correct. The rest were the lab questions themselves. Upon investigating those three categories, no other categories or subcategories emerged. Then I thought about the non-mathematical questions. I noticed that some were about what the lab question meant or about when the lab was due. There were no other types of questions that emerged from the data. Therefore, no further coding adjustments were necessary after reviewing the coded data one final time. Table 6 contains an excerpt from the data and my approach with coding the questions.

Table 6

Coding Online Questions

Code	Comment
	Tyson: 1. y = 5(x^2 + 1)^3
	y' = 5(3)(x^2 + 1)^2(2x)
	y' = 30x(x^2 + 1)^2
	y = x(x^2 + 1)^3
SM	Are you supposed to use [the product] rule here?
	y' = [x(3)(x^2 + 1)^2 (2x)] + [(x^2 + 1)^3 (1)]
	y' = 6x^2(x^2 + 1)^2 + (x^2 + 1)^3
	y' = (x^2 + 1)^2 [6x^2 + (x^2 + 1)]
	y' = (x^2 + 1)^2 (8x^2 + 1)
	Daniel: Okay, that looks good.
	I won't post my work because it is the exact same, but I got those answers.
PC	So now, do you think we just say that they both had to use the chain rule to solve for the derivative, but the second equation also had to use the product rule?
	That seems like how they are similar and different, but I don't really understand what it is asking for.
	Tyson: "They both had to use the chain rule to solve for the derivative, but the second equation also had to use the product rule," seems good to me.
	I don't really get what they're asking either.

Note. PC = Problem Clarification Question; SM = Specific Mathematical Question.

For the face-to-face environment, I found that I had to go through that data more times than the online data because it was quickly apparent that there were more types of questions asked. It was also evident that there were very similar types of questions as the online environment. I coded those questions using similar codes as those from the online environment. I looked at the questions that remained. I noticed that many questions were asked to specific group members. I coded these as questions for specific people. The rest of the questions were categorized according to their specific type, and were few in number. Table 7 contains an excerpt from the data parsed out into comments and my approach with coding the questions.

Table 7

Coding Face-to-Face Questions

Code	Comment
	Jayden: John's average speed.
PG	Why don't we just find the…?
	Josh: We're doing the $f(a)$ minus—
SM	*Kevin*: Would it be from 0 to 2?
	Jayden: Yeah, from 0 seconds to 2 seconds.
	Gary: Okay, so—
MQ	*Jayden*: And then what's the x-value?
MQ	How many feet?
SM	Sixty-four?
	0 minus 64…
	So it would be a 1…32 feet for every 1 second.
	Kevin: Cause he fell 64 feet in 2 seconds.
	Jayden: Yeah.

Note. MQ = General Mathematical Question; PG = Proposition for Group; SM = Specific Mathematical Question.

Using a Variety of Tools

There was specific literature on this aspect (NCTM, 1991), which I took as the lens for analyzing the data. I looked for any comment that could be construed as tool use, either tangible or intangible. I wrote down all the categories that were outlined in the framework. Then I explored the data sets from each of the environments and investigated how the students used each tool. The categories given in NCTM (1991) were: (1) any type of calculator use, (2) concrete materials, (3) pictures, (4) diagrams, (5) tables, (6) graphs, (7) invented terms and symbols, (8) conventional terms and symbols, (9) metaphors, (10) analyses, (11) stories, (12) written hypotheses, (13) explanations, (14) arguments, (15) oral presentations, and (16) dramatizations.

Going through the online data, I found that some of the suggested tools to enhance discourse were not present. I also found that there were some new types of tools, too. Therefore, I used methods similar to the constant comparative method to adjust the categories based on what I found in the data and the idea to look for additional tools. I found that even though the teacher was not listed as a tool in the literature, there was evidence that the online students used my comments as a tool for discourse. Additionally, the calculator was only used for computations in the online environment and did not fulfill other roles as outlined in Doerr and Zanger (2000). Explanations and arguments were combined into one category because all the explanations given were used as arguments. Finally, I noticed that the original lab questions were used as a tool to frame the comments in their posts. Table 8 contains an excerpt from the data parsed out into comments and my approach with coding the tool use.

Going through the face-to-face data, I also found that many of the suggested tools in the literature did not occur as in the online environment. In addition, I found that there were some new types of tools not mentioned in the literature, as was the case for the online environment. Again, I made some adjustments on the categories based on the data from the face-to-face environment as a result of using a constant comparative-like method. Like the online environment, explanations used by the students were their arguments. I could not track how often students used invented and conventional symbols because their written work was not available to me.

Table 8

Coding Online Tool Use

Code	Comment
	Rob: Bethany! You're way diligent at working these labs. Quinn, it was cool to sit by you in class today. So, since life is crazy and I'm way behind, I wanted to try to contribute and just add comments after examining both of your answers. 1. I agree with Bethany.
WEQ, CTS	2a. I agree with Bethany, and was wondering if it was necessary to factor a $(x^2+2x+1)^2$ out of the top or if that would just jumble it up more.
T ITS	2b. I also tend to agree with Bethany, but had a question for Shawn. Would you instead work by taking the chain rule of the entire parenthesis, like 3(the function)^2 and multiply that by the quotient derivative of the entire interior of the parenthesis?...

Note. CTS = Conventional Term or Symbol; ITS = Invented Term or Symbol; T = Teacher; WEQ = WebEQ.

The students in this environment also used the teacher as a tool and the original lab questions as a tool to facilitate their discourse. Table 9 contains an excerpt from the data parsed out into comments and my approach with coding the questions.

Initiating Problems and Making and Investigating Conjectures and Solutions

I took the final aspect of discourse and noticed that there were two main ideas that can be looked at separately: (1) initiating problems and (2) making and investigating conjectures and solutions. I found that this was a good division as I studied the data because the students would initiate problems in several ways. Then they would take these initial ideas and make and investigate conjectures based on the initial ideas.

Table 9

Coding Face-to-Face Tool Use

Code	Comment
OLQ	*Kevin*: …What time did the parachute pop out? *Jayden*: I would probably say 11 seconds. *Ashley*: I think we could say 11 seconds. *Kevin*: Yeah 11 seconds.
G	*Ashley*: [referring to graph] Wait, but then doesn't it start falling faster after 11 seconds? *Kevin*: No, it's up, the velocity starts going back up. *Ashley*: Oh, because it's going backwards?
HM	*Kevin*: [motioning with his hand] It's falling at a certain speed, but then it slows down. *Ashley*: Okay.

Note. G = Graph reference; HM = Hand Motions use; OLQ = Original Lab Question.

Initiating Problems

I read through the online data several times looking to see how the students began each problem in each task. I found that the resulting discussions were affected most by how many problems students initiated in the first group post rather than the content of their initial conjectures, which was what had influenced the face-to-face discussions the most. I tracked the different ways students initiated their labs for the group and then investigated the effects of each type of setup, which are illustrated in the next chapter. Initially, I had separated out the times when students posted their conjecture or solution to one, two, several, and all problem(s) at a time. There ended up being only a few times when students posted two problems at a time and the resulting structure of the discussion threads was very similar to the structure of when students posted several problems at a time. Therefore, I combined the category of posting answers to two problems at a time with several problems at a time. Table 10 shows an example of how I tracked the ways students initiated problems online for Lab 4 (see Appendix C).

Table 10

Tracking How Students Initiated Problems Online

Group Number	Tracked Initiations	Frequency
Group 1:	Kacy posted 1-6	In one post
	Jeremy posted 7-10	In one post
Group 2:	Daniel posted 1-8	In one post per problem
	Tyson posted 9-13	In one post per problem
Group 3:	Bethany posted all lab [some questions, but mostly conjectures]	In one post
Group 4:	Misty posted 1-5	In one post per problem
Group 5:	Jenny posted all lab [mostly conjectures]	In one post

Upon reading the data from the face-to-face section, it was apparent that they worked on each problem one at a time, in the sequence given. Therefore, when I looked at the data for how they started each problem, what they said stood out more than how their conversation ended up being structured. While reviewing the discussion around the beginning of each problem in the labs, I found that the students talked about the mathematics of the problem or they did not. The mathematical initiations were broken up into mathematical questions or mathematical propositions or conjectures, depending on if the students did so or not. The rest of the initiations were found to be either reading the question that the group was working on out loud or proposing a plan on how to approach the lab question. Table 11 shows how I coded a portion of the face-to-face data.

Making and Investigating Conjectures and Solutions

I read through the online data again looking for evidence of when students presented a conjecture or solution then how they investigated it further.

Table 11

Coding How Students Initiated Problems Face-to-Face

Code	Question	Comment
MQ	1	*Kevin*: Do we need to set it equal to zero and solve? *Gary*: Let's see. *Kacy*: That sounds really good. ⋮
GP	2	*Jayden*: Why don't we just find the… ? *Josh*: We're doing the f of a minus— *Kevin*: Would it be from 0 to 2? ⋮
RQ	3	*Jayden*: What was John's average speed during the last second before he hit the water? *Gary*: Thirty-two feet per second. *Kevin*: So was that—? ⋮
RQ	4	*Gary*: Okay. John's average speed during the last half-second. Or, is that just the one-half? Or to the—? *Jayden*: Well, it would be point 5 squared. *Gary*: Yeah point 5 squared. ⋮
MQ	5	*Kevin*: So, it would be 1.92 right? *Jayden*: Yeah. *Gary*: [talking to himself] Equals 6.24 over— ⋮
MP	6	*Gary*: The instant he hit below the water. So that's 2 seconds. The same equation. Sixty-four minus 16 [pause] squared [pause] so 4 times 16 [pause]. *Josh*: Oh shoot. What was John's speed at the instant he hit the water? So, I think I'll do the instantaneous part. I mean we can't plug it in like we did before, because it just gives us zero up front. ⋮

Note. MP = Mathematical Proposal; MQ = Mathematical Question; RQ = Reads Question directly from lab; GP = Group Proposal.

As I had done before with the other aspects, I noticed patterns and noted them down. The first pattern I noticed was that when a conjecture or solution was presented, the subsequent comments either stated agreement, disagreement, or a new conjecture entirely. However, most were agreements about what the student initially posted. There were no other identifiable patterns among the new conjectures, so I turned my attention to the comments that either agreed or disagreed with a conjecture or solution. The most apparent characteristic of the investigations that agreed with the initial conjectures was that there was no mathematical evidence given to back their claim. Therefore, I separated this category into statements that showed mathematical evidence and those that did not. I did the same with the category of comments that disagreed with the initial conjecture. Table 12 shows how a portion of the online data was coded in the manner I described previously.

Table 12

Coding of How Students Made and Investigated Conjectures and Solutions Online

Code	Comment
C	*Tyson*: Average for last second - I'm really not sure here: From 1 we know there are 2 seconds, so it is also the first second: h'(t) = –32t h'(t) = –32(1) h'(t) = –32 32 ft/sec, but I don't like the negative. Food for thought - here is another group: If t = 1, then his height at one second was 48 feet above the water. Then, (48 – 0)/(1 – 0) = 48 feet per second ⋮
AwM	*Mike*: I think that the right answer would be 48. I was looking over the information from other grops [sic] and it seems that the second equation is the one to use.
AwM	*Autumn*: I got 48 too. This is how I solved it: h(t) = 64 – 16(1)^2 = 48 ft average speed = ft/sec = 48ft/1 sec = 48ft/sec

Note. C = Conjecture; AwM = Agreement with Mathematical evidence.

I proceeded to use the same method for finding patterns in the face-to-face data. It was interesting that the same forces were at work as in the online environment, but the face-to-face environment had two different qualities that were not in the online environment. Thus, I gave them their own category. Consequently, the online environment had a quality not found face-to-face which I previously assigned its own category. Table 13 shows a portion of the face-to-face data and how it was coded.

Table 13

Coding of How Students Made and Investigated Conjectures and Solutions

Code	Comment
C	*Gary*: You did 3, you did 3:54.
E	*Jayden*: Yeah, because I did 9.9, which would be one 10th of an hour, which is 6 minutes, so—
	⋮
Aw/oM	*Gary*: Three thousand four hundred twenty-eight.
CL	*Jayden*: Uh huh, 3428.71.
Aw/oM	*Kim*: Uh-huh.
CL	*Kevin*: That's for 3:54?
E	*Jayden*: Uh huh, cause I just did 9.9 with is a 10th of an hour.

Note. Aw/oM = Agree without Mathematical explanation; C = Conjecture; CL = Clarification; E = Explanation.

Once I had the categories outlined based off of the aspects of discourse, and the data coded according to those categories, the results and conclusions based on the analysis were ready to be disseminated. It is important to note that the categories were not mutually exclusive. Some comments were categorized in more than one category based on the fact that their content could be seen in two different ways.

CHAPTER 5: RESULTS

In this chapter, I will present the results of the data analysis and thus illustrate how the three aspects of discourse, as outlined in the framework, played out in the online and face-to-face environments. Through these results, I will characterize the mathematical discourse in the online environment then the face-to-face environment in order to answer the first part of my research question. This detailed characterization will be followed up with a general summary of the results.

Online Student Mathematical Discourse

Listening To, Responding To, and Questioning One Another

Listening and Responding to One Another

The students in the online environment listened and responded to each other well despite the fact that the discourse was asynchronous. Nearly every time a proposed solution was posted, the other group members gave it feedback. Nearly every question was answered. An example to illustrate the usual responses that were given to a posting came from Group 2's work on the beach problem from Lab 2, Part B, Question 2 (see Appendix A).

The beach problem stated that one day, a cliff diver named John invited five friends to watch him in a cliff diving competition that started at 4:00 p.m. He and his friends arrived at the beach at 6:00 a.m. for John to practice and for his friends to watch. Over the course of the day, the population on the beach, which started with John's five friends, doubled every hour until the competition started. The second question in this problem asked on average how many people arrived per hour. From the first question, the group found that there would be 5120 people arriving at the time the competition started and that the equation $f(x) = 5 \cdot 2^x$ modeled the situation.

> *Daniel*: I thought I might take a shot in the dark at 2B by just using the average rate of change formula f(b) − f(a)/(b − a) and using 10 and 0 as the numbers. That answer would be 511.5 people per hour, but I really don't know.
>
> ⋮
>
> *Bill*: That's what I got as well, 511.5.

Evidence that Bill listened to Daniel's post came from his general comment that he got the same answer that Daniel did. This agreement meant that he must have read the post and noticed that Daniel's answer was 511.5 people per hour. He acknowledged to Daniel that he did the mathematics when he said, "That's what I got." However, we cannot know what mathematical approach Bill used because he was not explicit about one in his post, only that he got the same answer. For Bill, it was sufficient to forgo the explanation because the standing comment accomplished his goal of contributing to the conversation.

This example shows that the students in the online environment normally posted comments with just enough information to let the group know what work they had done with the lab. Most students listened to whoever made the first post of the problem, then responded, as Bill did, that they had also worked out the problem beforehand, and whether or not they received the same answer. If they did not receive the same answer, the students were usually good about responding and pointing out any discrepancies.

Questioning One Another

At times, students in the online environment asked each other questions as they conversed about the mathematics. Twenty-nine percent of the posts online contained questions made by the students. The types of questions they asked each other varied.

Table 14 sets forth the types of questions asked and an example of each. Figure 4 shows the frequency of the types of questions asked.

Table 14

Types of Questions Students Asked in Their Online Discussions

Question Type	Example Question
Lab Questions	(See Appendixes A, B, C)
General Feedback	"Does anyone have any comments on my work?"
Specific Mathematical Feedback	"Do we use limits here in this step?"
Course Requirements	"When is this lab due?"
Problem Clarification	"What's this problem asking for?"

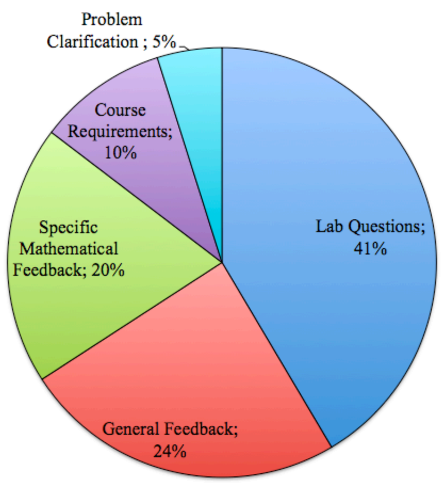


Figure 4. The various ways and frequency students asked questions in their online discussions[4].

The three most frequent types of questions that students asked online comprised the majority of the data in this category (85%). They were: (1) restating the lab questions,

[4] Percentages were calculated using the number of the certain question type divided by the total number of questions and not the total number of posts.

(2) questions that requested general feedback on an answer, and (3) questions about whether or not the specific mathematical procedure used was correct.

Restating lab questions. The most frequent type of question that was asked online came from students repeating the lab questions in their posts followed by their answer. These types of questions were posed to the student themselves, not to another student in the group. The manner these questions were used was in contrast with the rest of the types of questions used in their discussions, which were meant for the group. Their function in the discussion played out like tools in this environment. Therefore, they will be discussed as a tool later on. The next two most frequent types of questions had one thing in common. They occurred after a student posted a conjecture and needed feedback from the group on it.

Questions for general feedback. As students discussed mathematics in the online environment, 24% of the questions asked were requests for general feedback on a post the student just made. They typically asked for several things: (1) ideas on what to do next, (2) comments on the quality of their post, (3) general help with their post, (4) other general thoughts about their post, or (5) mathematical verification of the contents of their post. All of these requests were for general group input, nothing specific.

An example of this type of question was one that Jeremy asked when he posted a batch of answers as a contribution to the group discussion. He was from Group 1 and was working on Lab 2, Part A (see Appendix A). Part A of Lab 2 featured a different problem about our diver named John, who was mentioned previously. The students were to use a given formula that modeled John's height versus time over the course of one of his cliff jumps. The students calculated his average speeds over certain time intervals of his dive off the cliff. Jeremy posted his answers to Questions 2-6, and then asked the group a question for feedback:

> *Jeremy*: …A.6 [asking about the speed at instant he hit the water]
> $-32t = -32(2) = 64$ feet per second
> This is what I got for problem A. Most other groups have simular [sic] answers. The only one that I'm unsure of is number A.6. Any suggestions?

Jeremy asked the group for feedback on his answer to Question 6. (Questions 2-5 dealt with finding average speeds, which he did correctly.) The process to find an instantaneous speed was new for the students and very different than finding an average speed. Even though he answered Question 6 correctly, he wanted to be sure that he had correctly applied the mathematics by getting some general feedback from his group. Note that he did not ask about the mathematics behind the answer, the procedures he performed, or the numbers he used. The students who were requesting general feedback did so because they were unsure of the procedures in their post. Thus, their mathematical understanding was not as developed as those who asked for specific feedback or none at all.

Questions for specific mathematical feedback. The next most frequent type of question students asked online was about whether they should have used a specific mathematical idea in their answer to a lab question. There were many times when they had used a certain mathematical procedure in their solution, but were not completely sure they should have used it. The following example illustrates this. Here, Tyson, from Group 4, was working on Lab 3 (see Appendix B). Lab 3 required the students to take derivatives of similar functions and then compare the derivative processes in the end. This post showed his work on Question 1:

Tyson: 1[A]. y = 5(x^2 + 1)^3
y' = 5(3)(x^2 + 1)^2 (2x)
y' = 30x(x^2 + 1)^2
[1B.] y = x(x^2 + 1)^3
Are you supposed to use [the product] rule here?
y' = [x(3)(x^2 + 1)^2 (2x)] + [(x^2 + 1)^3 (1)]
y' = 6x^2(x^2 + 1)^2 + (x^2 + 1)^3
y' = (x^2 + 1)^2 [6x^2 + (x^2 + 1)]
y' = (x^2 + 1)^2 (8x^2 + 1)

Tyson did the first derivative correctly, perhaps without any problem. However, when he came to the second derivative, he was not completely sure what to do. His thought was to use the product rule. He went ahead and finished the problem using the product rule as shown in the rest of the post, following his inclination.

When students asked for specific feedback, it showed that they had a better understanding of the mathematics that those students who would ask for general feedback. They were sure about everything except the one specific concept they are asking about. The students who asked the group for general feedback were not sure about their entire answer. This shows that there were times that students working in the online environment did not have a sure understanding of the mathematics to a certain degree. It remains to be seen how this played out face-to-face.

The other types of questions students asked online were about: (1) course requirements and (2) problem clarifications. These types of questions were about directions for the students. Fifteen percent of the time they asked questions to clarify the problem directions and asked each other about the teacher's directions for what he wanted them to do or when to turn things in.

Questions on course requirements. The next most frequent type of questions students asked online was about course related matters. These were question like, "Is this lab due today?" Questions such as these did not affect their mathematical discourse and did not elicit mathematical explanations.

Questions on problem clarifications. There were times when students in the online environment had questions for clarification on what the problems in the labs were asking. For example, Lab 3 (see Appendix B) directed the students to take the derivatives of two similar functions. Then it asked, "For each pair, describe how they are alike, and how they are different." Group 2's initial thoughts with Question 1 on this particularly unclear requirement reflected the types of questions students had in this area:

> *Daniel*: So now do you think we just say that they both had to use the chain rule to solve for the derivative, but the second equation also had to use the product rule? That seems like how they are similar and different, but I don't really understand what it is asking for.
> *Tyson*: "They both had to use the chain rule to solve for the derivative, but the second equation also had to use the product rule" seems good to me. I don't really get what they're asking either.
> *Autumn*: I'm glad to see I'm not the only one confused.
> *Mike*: I agree with what the answer is and I think that they just want us to say what rules were similar in both which would be the product rule and chain rule.

From this excerpt we can see a slight difference in the type of question and resulting activity that occurred compared to what happens when the students requested general or specific feedback from the group. The first two types of questions came from posts or ideas where the students knew the direction to go. Here, the students had to take what they knew from their experience (of just having taken derivatives of two similar functions), decide as a group what the unclear directions meant, find the path to an acceptable answer, and then follow through with it. This type of negotiation reflected a typical discussion where students constructed their knowledge as a result of social interaction because they worked out and agreed on what the directions should mean and what the resulting mathematical arguments should be.

Using a Variety of Tools

Students discussing mathematics in the online environment were seen to use some sort of tool to facilitate discourse in nearly every one of their posts. There were nine different types of tool use found in their discussions. They were using: (1) conventional terms and symbols, (2) invented terms and symbols, (3) explanations and arguments, (4) original lab questions to organize a post, (5) other groups' ideas for consideration, (6) Microsoft Word attachments to the posts to communicate mathematics, (7) WebEQ to communicate mathematics, (8) the teacher to help with questions, and (9) calculators for computations. Figure 5 shows the frequency of the types of tools discussed in the online discourse.

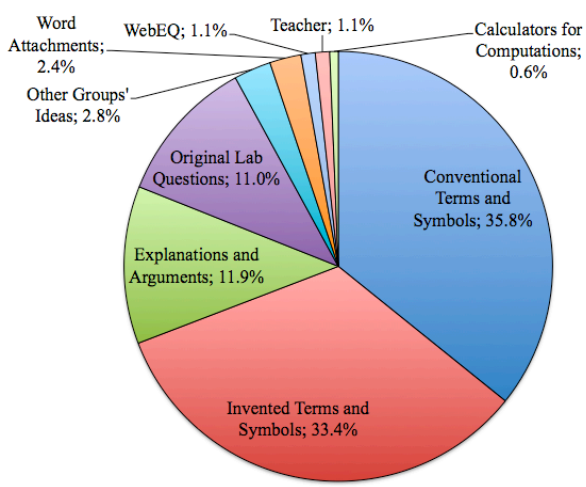


Figure 5. The various ways and frequency students used tools in their online discussions.[5]

[5] Percentages were calculated using the number of the comments about the certain tool divided by the total number of comments on tools and not the entire discussion.

Conventional and Invented Terms and Symbols

The two tools that were used most frequently, about 69% of the time in online discussions, were conventional and invented terms and symbols. The students used these terms and symbols in order to facilitate their discussions of mathematical ideas among one another. The most frequent means students used to communicate the mathematics was through plain text. The students' use of plain text had certain advantages and disadvantages. For example, it was advantageous to type out short invented expressions like "8x^2" or "1/2" as text because they could be typed quickly and could be interpreted just as quickly as the conventional "$8x^2$" or "$\frac{1}{2}$". However, when the students desired to communicate longer mathematical expressions, then typing the mathematics out using invented text became confusing for them. This preference was evident through the example of Group 2's work on the derivative from Lab 3, Question 2, Part A (see Appendix B). Here, Tyson took the derivative of the first function and meticulously showed its simplification step-by-step:

> *Tyson*: y = (x^2 + 2x + 1)^3 / 3x^2 + 1
> y' = [(3x^2 + 1)(3(x^2 + 2x + 1)^2(2x + 2))] − [(x^2 + 2x + 1)^3(6x)]
> (3x^2 + 1)^2
> y' = 3(x^2 + 2x + 1)^2[(3x^2 + 1)(2x + 2) − 2x(x^2 + 2x + 1)]
> (3x^2 + 1)^2
> y' = 3(x^2 + 2x + 1)^2[6x^3 + 6x^2 + 2x + 2 − 2x^3 − 4x^2 − 2x]
> (3x^2+1)^2
> y' = 3(x^2 + 2x + 1)^2[4x^3 + 2x^2 + 2]
> (3x^2 + 1)^2
> y' = 6(x^2 + 2x + 1)^2(2x^3 + x^2 + 1)
> (3x^2 + 1)^2
> Sorry, but "(3x^2 + 1)^2" is supposed to be the denominator the entire time, I hope it makes sense.
>
> *Autumn*: Yeah, it took me some time to figure out what exactly was going on, but it looks good from what I can tell. Awesome work!

Tyson's work was correct, but because of how it was written, other students had difficulty understanding and keeping track of what Tyson did in each step. In order to follow the discussion, the reader would have to translate the invented typed mathematical text into a conventional and readable form, either mentally or on paper. Then they would be able to evaluate its correctness. They would have to make sure that they wrote "(3x^2 + 1)^2" in the denominator the whole time, just as Tyson emphasized. This process takes time and would slow down the discussion flow for the reader. In Autumn's post, which followed Tyson's solution, this difficulty was illustrated when she did admit that it took time for her to figure out what Tyson was doing in each step. Even Tyson saw that it might be hard to read because he specifically stated that "(3x^2 + 1)^2" was the denominator to the entire expression the whole time. He also stated that he hoped it would make sense, too. Had Tyson used one of the options especially made for typing out mathematical equations conventionally, his post would have appeared considerably more readable (see Figure 6).

Students used invented characters to type out some of their mathematical expressions. Most of the invented symbols were adaptations of conventional calculator symbols understood by graphing calculators. The equations on the right side of Figure 6 are examples of this. Students used a caret (^) to signify that what comes after is exponential, an apostrophe (') to indicate derivative or something prime, and a backslash (/) for division. In addition, there were other types of invented terms used. For example, the students used the word "root" to indicate what came after in the parentheses was under a radical sign (√). They also used "inf" and "neg inf" to represent infinity (∞) and negative infinity (−∞), respectively.

$y=\frac{\left(x^2+2x+1\right)^3}{3x^2+1}$	=	y = (x^2 + 2x + 1)^3 / 3x^2 + 1
$y'=\frac{\left\{\left(3x^2+1\right)\left[3\left(x^2+2x+1\right)^2(2x+2)\right]\right\}-\left[\left(x^2+2x+1\right)^3(6x)\right]}{\left(3x^2+1\right)^2}$	=	y' = [(3x^2 + 1)(3(x^2 + 2x + 1)^2(2x + 2))] − [(x^2 + 2x + 1)^3(6x)] (3x^2 + 1)^2
$y'=\frac{3\left(x^2+2x+1\right)^2\left[\left(3x^2+1\right)\left(2x+2\right)-2x\left(x^2+2x+1\right)\right]}{\left(3x^2+1\right)^2}$	=	y' = 3(x^2 + 2x + 1)^2[(3x^2 + 1)(2x + 2) − 2x(x^2 + 2x + 1)] (3x^2 + 1)^2
$y'=\frac{3\left(x^2+2x+1\right)^2\left[6x^3+6x^2+2x+2-2x^3-4x^2-2x\right]}{\left(3x^2+1\right)^2}$	=	y' = 3(x^2 + 2x + 1)^2[6x^3 + 6x^2 + 2x + 2 − 2x^3 − 4x^2 − 2x] (3x^2+1)^2
$y'=\frac{3\left(x^2+2x+1\right)^2\left[4x^3+4x^2+2\right]}{\left(3x^2+1\right)^2}$	=	y' = 3(x^2 + 2x + 1)^2[4x^3 + 2x^2 + 2] (3x^2 + 1)^2
$y'=\frac{6\left(x^2+2x+1\right)^2\left(2x^3+x^2+1\right)}{\left(3x^2+1\right)^2}$	=	y' = 6(x^2 + 2x + 1)^2(2x^3 + x^2 + 1) (3x^2 + 1)^2

Figure 6. A comparison of equations from Tyson's work in conventional type versus invented plain text.

Explanations and Arguments

The next most frequent type of tool used in student online discussions was explanations and arguments. These explanations were given about 12% of the time tools were used. When used, explanations were an effective tool for students to help one another to understand and defend their mathematical ideas. When students were given an opportunity to understand and defend their mathematical ideas, only at certain times did they take advantage of such opportunities and use them. It was recommended by the literature that this tool of explanation and argument be the one that enriches mathematical discussion (e.g., Ball, 1993; Lampert, 1990; Yackel & Cobb, 1996). There were three main ways students explained and argued the mathematics. They would: (1) use only a verbal explanation for their arguments, (2) use only numbers and operations, and (3) use both operations and verbal explanations to convince one another of correct arguments.

Explanation only. When students only used explanations for their arguments as tools in their discussions, most of the time they posted explanations without showing the mathematical operations that went into the answers. This type of explanation is exemplified by a selection that comes from Group 3's work on the diving problem from Lab 2, Part A, Question 6 (see Appendix A). It began with Bethany attempting to determine John's speed at the instant he hit the water from his dive:

> *Bethany*: 6. What was John's speed at the instant he hit the water? How do you know?
> t = 2
> h(t) = 64 − 16(2)^2
> = 0 ft
> speed = 0 ft/s
> ⋮
>
> *Rob*: On number 6, I think if we input numbers approaching closer and closer to 0 we will find the instantaneous velocity. Therefore,
> t=1.99 leads to 63.84 ft/sec
> t=1.999 leads to 63.984 ft/sec
> t=1.9999 leads to 63.9984 ft/sec
> Because of that, as John approaches closer and closer to the water, he is approaching 64 ft/sec, which is his instantaneous velocity, or his speed at the instant he hit the water.

In, Questions 2-5, Bethany used the average rate of change to determine John's speed over the whole two second dive, over the last second, over the last half-second, and finally over the last tenth-of-a-second. In Question 6, she used the same formula and arrived at zero for the instantaneous rate of change. (Correct computations would actually give her an indeterminate number, 0/0). Rob then posted his answer without specifically mentioning where Bethany made her mistake. He explained that a limiting process would help them find an answer that was not in an indeterminate form. He then used examples of average rates whose time intervals were getting smaller and smaller. Finally, he was able to extrapolate the speed John was going when he hit the water based on the limiting process. Rob did not show the operations performed using the time indices and how they lead him to his answer. However, he explained the answer fairly well when he said that as John got closer and closer to the water (i.e., time approaching 2 seconds and resulting velocities approaching 64 ft/sec), the instantaneous velocity would actually be the 64 ft/sec.

We can see from this example that the reader would probably be convinced of Rob's explanation. However, they would be even more convinced had Rob shown them the mathematical operations he used to come up with the numbers. Because he did not, the mathematical evidence should have been verified by the reader in order to be absolutely sure that the post was mathematically correct. In this case, the reader would need to plug 1.99, 1.999, and 1.9999 into the average rate formula and see if the resulting rates were 63.84, 63.984, and 63.9984 ft/sec, respectively. Therefore, if the student was not thorough in their explanations, the resulting discussion for the reader would flow okay at best.

Operations only. Another way students used mathematical evidence to show and convince one another of correct argument was to convince their group members of the correct answer using only mathematical operations. An example of this type of explanation is found in Group 4's work on the beach problem from Lab 2, Question 2. Max showed his group how he calculated the average number of people that arrive to the beach each hour, "#2 5120 − 5 = 5115 divided by 10 which equals 511.5 people per hour." Here, Max was finding an average rate. To do this, he subtracted the number of people that arrived at the beach in the beginning from the number of people that arrived at the time the competition started. Then he divided that by the number of hours that elapsed since the first people arrived until the time the competition started.

The reader would probably be convinced that this is the correct answer. However, their conviction would not be as strong as it could have been if the student briefly explained where the 5120, the 5, the subtraction, the 10, and the division came from. As in the previous example, the reader would had to have taken a moment to make sure the numbers and operations given in the post had been used correctly.

Operations and explanation. The final way online students used mathematical evidence to show and convince one another of correct argument was to include the mathematical operations along with an explanation of the answer and where it came from in their post. A typical example of this approach was from Group 2's work on the diving problem from Lab 2, Question 4. This example is from Tyson's post of his interpretation of a method used by another group to solve the problem:

> *Tyson*: Again if t = 1.5, because he has traveled for that long already, then his height at half-a-second was 36 feet above the water.
> Then, (36 − 0)/(0.5 − 0) = 72 feet per second.

Tyson explained to a degree where his operations were coming from and how the numbers he used the solution, along with showing the mathematics. However, he did not mention how the numbers fit into the average rate formula and what the 72 ft/sec represented. This example is regarded as what students in the online environment usually did. They explained the mathematics and showed operations, but were never too explicit with either.

Original Lab Questions

The next most frequent tool used in students' online mathematical conversations was using the original lab questions to help organize the students' posts. They did this 11% of the time tools were used. Students would start their post by copying and pasting the lab question(s) into their message box and type out their answer after each one. It served as a good organizer to the post, thus allowing the reader to be able to know exactly what question each answer was referring to. As an example of how this was done, we use the first post of Group 3's work on the diving problem from Lab 2 (see Appendix A):

> *Rob*: 1. How many seconds after he started his dive did John hit the water?
> 0 = 64 − 16t^2
> −64 = −16t^2
> (Factor out a −1 from both sides so you can take the square root?)
> (−1)root 64 = (−1)root(16t^2)
> −8 = −4t
> time = 2 seconds
> 2. What was John's average speed as he fell from the top of the cliff into the Lake?
> Change in distance/Change in time
> (64 − 0)/(2 − 0) = 32 feet per second
> 3. What was John's average speed during the last second before he hit the water?
> If t = 1, then his height at one second was 48 feet above the water.
> Then, (48 − 0)/(1 − 0) = 48 feet per second
> 4. What was John's average speed during the last half-second before he hit the water?
> Again if t = 1.5 because he has traveled for that long already, then his height at half a second was 36 feet above the water.
> Then, (36 − 0)/(0.5 − 0) = 72 feet per second
> * Does this look right? I didn't want to go too far if my math was wrong.
> Thanks, - Rob

Here, Rob posted each question along with each answer for the first four questions in Lab 2. Each of his answers is understandable and we know how he got them. The understanding gained here for the reader is the reason posting the questions, or restating the question, with the answers is an effective tool for discourse. Because the reader gains this understanding, they are ready to give prompt and accurate feedback.

A contrasting example can be found in Jenny's work on the rocket problem from Lab 4, Group 5. She was the first from the group to post and posted answers to all the problems in the lab:

Jenny: 1. A little before 2 sec—because soon after that it starts to fall.
2. Around 2 sec—because that is where it changes from a positive velocity (upward) to negative velocity (downward).
3. Around 180ft/sec? Explanation?
4. Around 190ft/sec? Explanation?
5. 11 sec—because that is where it changes from negative velocity (downward) to positive velocity (upward).
6. The rockety [sic] was falling because you can see the velocity change the parachute made in the graph.
7. You don't know, do you?
8. Between 1 and 2 seconds—because that where the slope is the steepest.
9. Do you use change in velocity over the change in time to find this? well....its instantanious [sic] so....what do you use?
10a. 0-2 seconds—climing [sic] and speeding up.
b. I don't get the second part of these next few questions. Help?
11. I don't know. Help?
12. 2-11 sec—downward, speeding up?
13. I don't know. Help?

When the question from the lab was not included in the post, then the discussion was not easy to follow. Jenny asked for help in several problems, but if the reader was going over this post, they would not be able to immediately give that help. They would have to constantly refer back to a copy of the lab questions in order to assess the validity of her answers or know how best to answer her questions. When posting initial conjectures, it is beneficial to let the reader know what questions are being answered, if the he or she would like the best and quickest feedback.

Other Groups' Ideas

As students discussed mathematics online, they posted comments about the topic at hand using ideas found in other groups' discussions. The discussion board was set up so that any student could view any discussion thread at any time. Concrete evidence that students did this was sparse. It is possible, and likely, that students went to other discussion threads without mentioning it. Therefore, it was not possible to track the number of times they did this. An example of the kind of contribution to the discussion research in other groups' conversations can have is from Group 2's work on the diving problem from Lab 2, Part A, Questions 2-4 (see Appendix A). Tyson posted a couple of ideas from other groups about how to calculate John's average speed over the whole jump (Question 2), over the last second (Question 3), and over the last half-second (Question 4):

> *Tyson*: [Question 2]
> Here's what another group did:
> Change in distance/Change in time
> (64 − 0)/(2 − 0) = 32 feet per second…
> [Question 3]
> Food for thought—here is another group:
> If t = 1, then his height at one second was 48 feet above the water.
> Then, (48 − 0)/(1 − 0) = 48 feet per second…
> [Question 4]
> Here is how someone else went about doing these problems:…
> h(t) = 64 − 16(1.5)^2 = 28 ft
> av speed = 28 ft/0.5 s = 56 ft/s

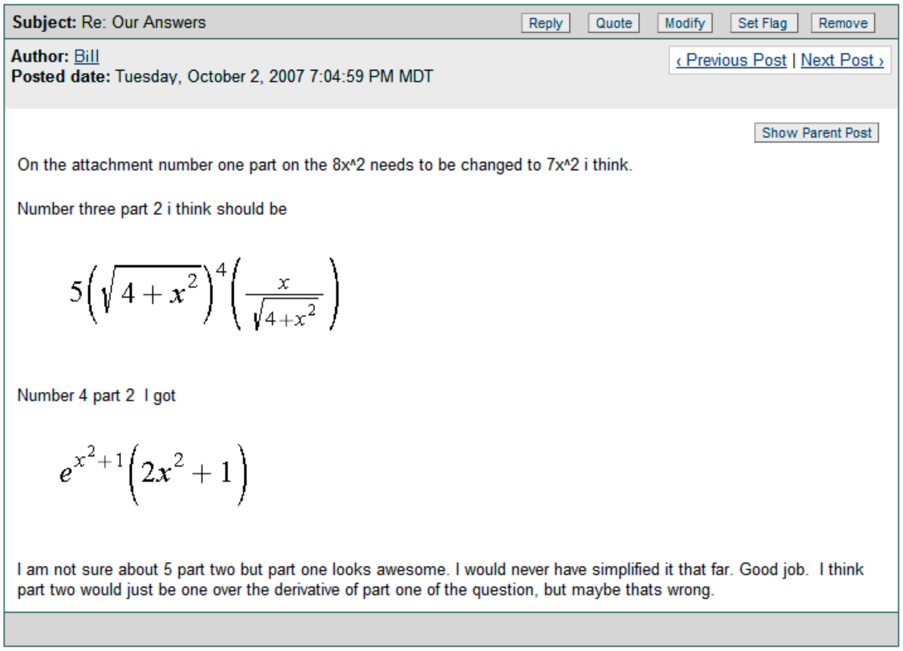
Because Tyson used multiple groups as a resource for his post, different approaches to the same problem type were used. Notice the way Questions 2 and 3 used $\frac{f(b)-f(a)}{b-a}$, the average rate of change formula over the interval from *a* to *b*. Contrast that with Question 4, which used a way of calculating the average rate of change by finding the average distance traveled over the time interval. These two strategies gave the group more to think about and assess. This example shows how resourceful students can be if there are multiple discussions of the same topic developing in the discussion board.

Attaching Word Documents to a Post

The first of two alternatives students used to sharing mathematical expressions without using plain text was to type out their equations using Microsoft's Equation Editor in Microsoft Word and then attaching that document to their post. The students used this a few times. When a student used Equation Editor, they were able to see the equation as they wanted it to look like throughout the whole process from typing to post. (This characteristic of Equation Editor is a contrast to how students were able to use the next tool, WebEQ.) It is likely that students had some experience with Equation Editor before, too. Therefore, they were more apt to use it to communicate large expressions than WebEQ. Students mostly used this feature when it was their turn to compile the group's final answers and submit them for grading.

WebEQ

The second alternative online students had to input mathematical expressions in their posts was to use a Java program especially designed to type mathematical expressions into a discussion board. This program is called WebEQ. It was used only a few times. The benefit to using this program was that you could type your mathematical expressions and insert them directly into your post. A good example that showed when to use WebEQ and when not to was from Group 2's work on the derivatives from Lab 3 (see Figure 7).



Subject: Re: Our Answers Reply Quote Modify Set Flag Remove

Author: Bill ‹ Previous Post | Next Post ›

Posted date: Tuesday, October 2, 2007 7:04:59 PM MDT

Show Parent Post

On the attachment number one part on the 8x^2 needs to be changed to 7x^2 i think.

Number three part 2 i think should be

$$5\left(\sqrt{4+x^2}\right)^4\left(\frac{x}{\sqrt{4+x^2}}\right)$$

Number 4 part 2 I got

$$e^{x^2+1}\left(2x^2+1\right)$$

I am not sure about 5 part two but part one looks awesome. I would never have simplified it that far. Good job. I think part two would just be one over the derivative of part one of the question, but maybe thats wrong.

Figure 7. Illustration of a post utilizing WebEQ mathematical writing program.

In the beginning of his post, Bill referred to a correction he saw that should have been made in the proposed final answer for his group. Notice that he just typed out "8x^2" and "7x^2" knowing that it was not necessary to put "$8x^2$" or "$7x^2$" in a special form. However, he then saw that the answers to a couple of derivatives were more complicated and used WebEQ to communicate them. In order to illustrate the contrast his work had with the usual way students opted to communicate their expressions, I created Figure 8.

$$5\left(\sqrt{4+x^2}\right)^4\left(\frac{x}{\sqrt{4+x^2}}\right) \quad = \quad \text{(5(root(4+x^2)))^4(x/root(4+x^2))}$$

$$e^{x^2+1}\left(2x^2+1\right) \quad = \quad \text{(e^(x^2+1))(2x^2+1)}$$

Figure 8. Comparison between expressions using WebEQ and text only.

Again, there is a difference in the time it takes to communicate the mathematics between reading and understanding expressions in WebEQ versus plain text. With regard to the typed mathematical expression, the reader would need to sort out the expression and convert it to something similar to the left side of the figure in order to see how all the terms related. The WebEQ form is easier to understand, and students can evaluate the expressions therein for correctness faster. One can quickly see that the first equation needs some more simplification, something not so transparent when it is typed out.

The Teacher as a Tool

As students participated in discussions about mathematics online, there were times when they felt that no one in their group could give them the feedback they needed. Additionally, if a search for ideas from other groups did not pan out, they posed their questions to the teacher, thus intending to use the teacher's responses as a tool to facilitate their discourse. This type of tool use happened only a few times. An example to illustrate how the students requested the teacher's assistance in their discussion was from Group 3's work on the derivatives from Lab 3, Questions 1-5. Bethany began by posting her answers to the entire lab. Then Rob posted some clarifications and comments about Bethany's post while asking the teacher some questions regarding her responses (see Figure 9).

Rob started by asking the teacher about a different, more conventional, approach to the solution of Question 2, Part B. Because his approach was different than Bethany's, he had some doubts about whether or not it was true. The group could have replied to the post and answered his question thus coming to a consensus, but they did not.

Subject: Re: Group Discussion Reply Quote Modify Set Flag Remove

Author: Rob
Posted date: Wednesday, October 3, 2007 4:49:31 PM MDT

‹ Previous Post | Next Post ›

2b. I also tend to agree with Bethany, but had a question for Shawn. Would you instead work by taking the chain rule of the entire parenthesis, like 3(the function)^2 and multiply that by the quotient derivative of the entire interior of the parenthesis?...

3b. I agree with Bethany, but Shawn, can it be simplified as $\frac{10x\left(\sqrt{4+x^2}\right)^4}{2\sqrt{4+x^2}}$ which would reduce again as 5x (the root) raised to 3?...

5b. I agree with Bethany, but was wondering if the ln in the denominator should be removed. Also, one more question for Shawn, similar to 2b. Instead, would you work the chain rule of the whole equation, ex. 3ln(equation)^2 and then multiply that by the derivative of ln(equation)

Figure 9. An example of the way the teacher was used as a tool online.

His next question to the teacher was also something the group could have handled, which was about simplifying a fraction. His final question for the teacher about the meaning of the second function in Question 5, was one that the teacher could have answered. The teacher had the option of stepping in and clarifying whether $\ln(x^2 + 2x + 1)^3$ meant one of two interpretations: (1) $\ln[(x^2 + 2x + 1)^3]$ or (2) $[\ln(x^2 + 2x + 1)]^3$ (see Appendix B). However, the teacher gave the group the chance to determine for themselves what options would result from either interpretation and decide as a group what their answer would be. This decision making could have played out well as it did another time the teacher let students decide what the directions meant about comparing the derivatives in Lab 3 (see "Questions on Problem Clarifications," p. 55). However, no one in Rob's group answered his question.

The Calculator as a Tool

Explicit evidence for the calculator being used in the online environment is little. Due to the way that some students posted their operations and numerical answers, it was probable that they used a calculator to find their answer. However, specific calculator use was usually not mentioned. There were a few occasions in which a student did talk about the way he or she used their calculator. This example is from Group 1's work on the derivatives from Lab 3, Question 1, Part B (see Appendix B). Jeremy had already attempted to make sense of the derivatives from the first question. He even used his calculator's graphing and table functions to help him, but could not come to any helpful conclusions for him. He used the calculator's symbolic derivative function to find the derivatives of the functions in the first question:

> *Jeremy*: I checked my answer for 1b with the derivative function on my graphing calculator.
> This [is] what I got:
> dx = (x^2 + 1)^2(7x^2 + 1) or
> completely expanded = 7x^6 + 15x^4 + 9x^2 + 1
> I am not sure if these are correct or if they take me farther away.

It seems that Jeremy was not too skilled with the derivative function at that time. He had the correct answer in two forms, but was unsure about them. It was very helpful for the discussion to have the other students see what Jeremy had found on his calculator. The context helped the students know where he was coming from and how to possibly fix his answer if it was incorrect. This calculator use could also have spurned other students to try and use their calculators and post their findings. However, the students did not end up discussing how the calculator could further help in this lab.

Initiating Problems and Making and Investigating Conjectures and Solutions

Initiating Problems

Students working in the online environment discussed how they initiated the mathematical tasks 34% of the time. They did this in several ways (see Figure 10) by: (1) answering one question per post, (2) answering several questions per post, and (3) answering all the questions in one post. The way that the students arranged their beginning posts in their online lab discussions affected the way subsequent posts were made which in turn affected overall discussion readability. When students initiated the lab problems, the content of these posts were usually proposed conjectures and solutions. They did not usually try to coordinate the initial approaches to the problems with their groups.

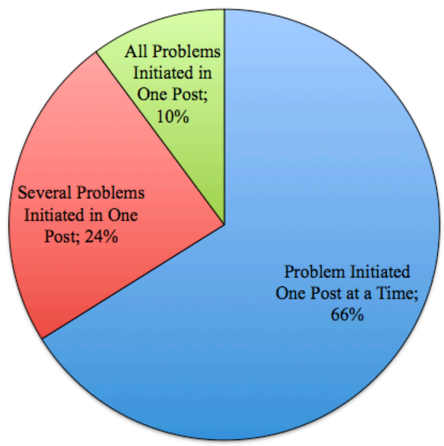


Figure 10. The various ways and frequency students initiated the lab problems in their online discussions[6].

One question per post. The most frequent way to initiate problems online was to make a separate post per problem. This post would contain a proposed conjecture or solution to the problem. In that way, subsequent posts could be made as replies to the initial post. When students did this, each question ended up having its own threaded discussion and thus the entire discussion related to the problem could have been easily read. This idea is illustrated in the following example of Group 2's work on the rocket problem from Lab 4, Questions 4-5. Lab 4 asked questions about the height and

[6] Percentages were calculated using the number of the certain initial post type divided by the total number of posts initiating problems and not the total number of posts.

acceleration of a rocket from the time it was launched, when it popped out a parachute, and until it hit the ground. The key to this lab was that the students were only given a graph which tracked the rocket's velocity through the whole flight (see Appendix C). Here is what they said:

> *Daniel*: [Question 4] I think the velocity would be zero when it is at its highest point because once it gains negative velocity it would be heading back toward the ground. It keeps going up until the velocity equals zero.
>
> > *Tyson*: That sounds good.
> >
> > > *Autumn*: Yes, that makes sense.
>
> *Daniel*: [Question 5] The parachute looks like it opened at 11 seconds because it would slow the rate that it was falling at and the graph "jumps" at eleven seconds. So that's what I would guess.
>
> > *Tyson*: The acceleration decreased and then began to level out, what one would expect from a parachute.
> >
> > > *Autumn*: Yes, I would think the answer is 11 seconds, because the spike on the graph and the sudden slowing of acceleration downwards.

Daniel had made two initial postings for his answers to Questions 4 and 5, one right after the other. Tyson came and replied to both posts. After, Autumn replied to Tyson's posts. Then the next reader, who would read them all, could read the discussion linearly, from top to bottom, and by so doing he or she would be able to easily follow all the comments made about each problem regardless of when they were posted. The skill to structure this type of discussion developed over time for the students. The groups did not do this much in Lab 2, but it became more prevalent in Labs 3 and 4. By this development and the fact that it is the most frequent way students initiated problems, we can see that students themselves discovered this as the most efficient way to initiate their problems and structure their online discussions. They noticed that the way the discussion appeared in the end would be based on how it began. The majority chose the one post per answer approach because in the end it was the most readable.

Several questions per post. The next most frequent way students initiated problems online was to make a post containing the answers to a small batch of problems. This approach was particularly used in Labs 2 and 3. Batch size reflected lab setup. Lab 2 consisted of two main problems, Part A, the diving problem with six questions and Part B, the beach problem with four. Students usually made one post for Part A and one post for Part B. (The exceptions to this rule featured students making posts one at a time or all at once.) Lab 3 consisted of five pairs of functions with which the students found the derivatives. Most of the posts featured answers to both of the functions for one problem. Lab 4 had 13 questions about the flight of the model rocket, and so the students usually posted answers to this lab one at a time.

Group 1's initial posts reflected the usual type of initiation for Lab 2 (see Appendix A). Jeremy made the first post with his answers to Part A. This excerpt from the data was given under the previous section entitled, "Question for General Feedback" (pp. 52-53). They generally talked about more than one problem at the same time. This type of discourse was slightly harder to follow because the reader had to go back to the original post to find which problem the reply post was referring to. (If the discussion was threaded with one question per post, then the question would not be hard to find if the reader needed to refer to it.) In order to go over the process the reader goes through when reading a discussion face-to-face with several questions per post, we refer to the reply to Jeremy's original post:

> *Kacy*: I agree with all of your answers. For 6, I also got 64, which makes sense because in all he is falling 64 feet every two seconds, so when he hits the water he should be going 64 ft/sec.

The reader, who would follow Kacy's post, would need to locate Jeremy's original post, search through it to find his response to Question 6, and read it to verify whether or not Kacy's response truly agreed with his. Kacy's post was an example of a typical response to initial posts that answered several lab questions at once. They usually addressed more than one idea developed in the initial post at the same time. It read as though one was reading several threads at once. Because students included more than one topic in a post, the resulting discourse did not end up being quite as organized as the situation in the previous section.

For most of the labs, the multiple-topic posts somewhat hindered discourse, however, for Lab 3 it was the best way to go. The lab required that students to post the derivatives of two similar functions and compare the processes or rules used to obtain them. Some students had done this effectively in one post. On the other hand, if they posted the derivative to the first function in one post and another student posted the derivative of the second in another, then the resulting discourse comparing the two derivatives would still have to be posted. This situation caused students to either forgo the explanation or insert it in a spot that hindered the flow of discourse.

All questions in a post. The last way students initiated problems for online discussion was to simply answer all the questions of the lab in one post and be done with it. When students did this, they felt that they had contributed enough to the discussion for that week and usually did not post again. It was then left up to the rest of the group to sort out the answers to determine validity and completeness. An example of the resulting discussion that followed such an initial post came from the work of Group 3 on the diving and beach problems from Lab 2 (see Appendix A). Bethany had posted all the answers to the lab. The subsequent comments went as follows:

> *Jared*: I found out that other groups think that the population was 5120 because they forgot to take into account that John was there as well, so we would start the equation with 6 as a coefficient…
>
> *Rob*: So I've been trying to figure out how we got different answers on questions number four, and I finally realized that 64 – 36 is 28. I've been assuming it's 36 the whole time, but that's definitely not right.
> On number 6, I think if we input numbers approaching closer and closer to 0 we will find the instantaneous velocity. Therefore,
> t=1.99 leads to 63.84 ft/sec
> t=1.999 leads to 63.984 ft/sec
> t=1.9999 leads to 63.9984 ft/sec
> Because of that, as John approaches closer and closer to the water, he is approaching 64 ft/sec, which is his instantaneous velocity, or his speed at the instant he hit the water.
>
> *Rob*: [in a different post] This is the same type of calculation question B # 3 is asking for—to pick a few values that are close to the finishing time (4 PM) and find the instantaneous rate of people arriving. To do this we'll need to pick times between 3 and 4 PM, mostly closer to 4. I would have attempted this myself, but I don't know what formula was used to find the values and I haven't figured out one yet myself.

In the first post, there was no mention of a problem number. In order to follow the discussion, the reader would need to search the entire contents of the lab answers from Bethany in order to find the question asking about the total population arriving the instant the competition began. Then it would be possible to determine whether or not Bethany included John in her calculations and decide if 6 people should have been used. The second post discussed discrepancies in Questions 4 and 6. The reader would need to go back to Question 4 in Bethany's initial post to verify that 28 was used and that the resulting operations were correct. Then the reader would definitely need to go back to Question 6 to read what Bethany did to generate such a detailed analysis from Rob. After, they would need to compare who was correct and post an agreement with Bethany's or Rob's answer. Finally, to continue to follow the conversation, the reader would need to go back to Bethany's post for Question 3 on Part B in order to see how she handled choosing the intervals. Then the reader would be able to determine if Rob's suggestion was correct or if Bethany's original answer was the one to use and advocate it. From these examples, we can see that this type of discourse would be the hardest to follow because the reader was constantly going back to the original post to catch up with the topics discussed in the subsequent posts.

From the examples on how students initiated problems, it was evident that the students' choices about how to approach a lab discussion in the online environment depended highly on how they initiated the problems of the lab in the discussion board. It is easiest to follow a discussion with one post per question. At times, posts may feature several topics, like the ones from Lab 3. However, they should not include too many otherwise the resulting discourse would become convoluted, as in the case of when all questions were answered at once.

Investigating Conjectures and Solutions

As students discussed mathematics online, 69% of the time they were found to be conversing about each others' conjectures and solutions. While thus discussing conjectures and solutions, 51% of the time they were further investigating certain conjectures and solutions. Only 29% of those investigations used mathematical evidence to support or refute the preliminary conjecture or solution. As students investigated a conjecture or solution, they: (1) posted that they agreed with the mathematical idea, (2) posted that they disagreed with the mathematical idea, or (3) posted their own mathematical idea (whether it be the same or different) without agreement or disagreement. Figure 11 shows the frequency of the ways students investigated conjectures online with or without mathematical evidence to support their claims.

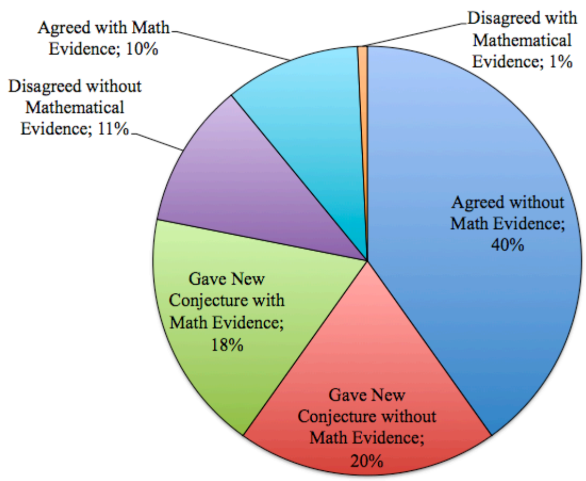


Figure 11. The various ways and frequency students investigated conjectures and solutions in their online discussions.[7]

Investigating conjectures and solutions without using mathematical evidence. The majority (71%) of the instances when students investigated conjectures/solutions did so without being explicit about the mathematics they used to do it. Forty percent of the investigations students did online stated that they agreed with the initial conjecture/solution without showing mathematical evidence. They would post single comments like, “I got the same answer!” or “Everything is correct for Parts A and B.” We must assume that since they stated that they had received the same answer or that they checked the other’s for correctness. However, if a student had received the same answer as one who had posted already, what choices do they have to contribute to the discussion? They can repost the same work or just simply state that they received the same answer. If they did find the answer a different way, then it is imperative, for

[7] Percentages were calculated using the number of the different types of investigated conjectures done divided by the total number of investigated conjectures and not the total number of posts.

discussion's sake, that they do post the mathematics along with the answer. By so doing, students would then have a chance to compare their various approaches to the solution. Thus, they can obtain a more concrete understanding of the topic as a whole.

The next largest category of how students investigated conjectures without mathematical evidence was when students posted a new conjecture, whether it be their own or another group's, as a response to the initial one. This type of post was done 20% of the time. They did so without stating they agreed or disagreed with the initial conjecture. Additionally, they did not mention the mathematics that went into their conjecture. They would post things like, "I think the answer is 48," without saying why. Another example would be, "2a. (−4x − 6x^2 + 2) / (3x^2 + 1)^2," again without evidence, and when someone already answered the first part of Question 2 from Lab 3 (see Appendix B).

The final category of work without showing mathematical evidence online was when students disagreed with the answer to the mathematical task. It is not possible that one can completely convince another of a contrasting answer if they do not show mathematically why the other person is wrong and/or why they are correct. Yet, students attempted to do so 11% of the time. An example of this was when a student found the derivative to a function from Lab 3, and the entire reply post stated, "Hey! I got almost the same answers as you but I got a 6 instead of an 18 in 1b, but I could be wrong." In this example, the student had a polynomial and the answer with a coefficient of 6 compared to the initial post, which contained the same polynomial, but with an 18 instead. The student investigating the conjecture would have served her group better if she was able to find and explain mathematically exactly how his 18 was incorrect and how her 6 was correct.

These examples show that online students are not prone to include mathematical evidence when they discuss mathematics online. There are occasions, as in the first example, where it might not be necessary to show the mathematics. On the other hand, there are times when one may have a different idea for an answer or when there is a disagreement, then it is very important to show mathematically what the answer should be.

Investigating conjectures and solutions with mathematical evidence. As stated previously, students used mathematical evidence to support their investigations of conjectures/solutions only about a third of the time. The occasions when they investigated conjectures/solutions with mathematical evidence are more in line with the purposes of student mathematical discourse according to the literature (e.g., see Ball, 1993; Lampert, 1990; Yackel & Cobb, 1996). It is beneficial for the discussion to show how you got your answers. The following examples will illustrate why. The most frequent type of investigation in this category was when students offered a new conjecture/solution for the group to consider in place of the old one. They did so without stating that they agreed or disagreed with the initial conjecture.

The posts that feature this lack of agreement or disagreement usually followed two types of initial posts. If students did not know how to further the conversation, they could have gone to different groups for ideas. A portion of the times when students proposed new conjectures consisted of ideas taken from the discussions of other groups (e.g., "Other Groups' Ideas," pp. 64-65). The posts consisted of the conjectures of other groups which were posted to help generate ideas for group discussion. Another possible reason students posted conjectures without following the discussion was because they did not know if they agreed or disagreed with the initial conjecture. Sometimes they had an idea that their contribution was related to the point of the conjecture, but not exactly how. For example, Misty read an initial conjecture from Group 4's work on the beach problem from Lab 2, Question 1 (see Appendix A). She was not sure if her investigation was correct:

> *Misty*: For #1 I got that there were 5120 people. I don't know if I did it right though. I just did it by hand so at 6 a.m. there's 5 people, 7 a.m. there's 10, 8 a.m. 20, 9 a.m. 40, 10 a.m. 80, 11 a.m. 160, 12 p.m. 320, 1 p.m. 640, 2 p.m. 1280, 3 p.m. 2560, and 4 p.m. 5120.

This excerpt is an example of a case where a new conjecture was made, using mathematical evidence, and the student was neither agreeing nor disagreeing with the initial post. She was just unsure of whether or not it was correct. This uncertainty shows that when students use mathematical evidence to investigate conjectures, sometimes they were not completely sure they were doing it correctly. But, because they showed evidence another student could assess validity.

Ten percent of the time students investigated conjectures/solutions, they demonstrated reasons why that they agreed with the initial conjecture. A good example of this was when Group 2 was working on the diving problem from Lab 2. Bill read the initial conjectures from the group from Part A. They used several different approaches to the problem. His post was one of agreement, but showed how to arrive at the answers in a consistent manner:

> *Bill*: That is the right way to do it.
> 1. time = 2 seconds
> 2. average speed = 64 ft/2 sec = 32ft/sec

3. height = 64 − 16(1)^2 = 48 ft above the water so 48/1 sec = 48 ft/sec
4. height = 64 − 16(1.5)^2 = 28 ft above the water so 28/.5 sec = 56ft/sec
5. height = 64 − 16(1.9)^2 = 6.24 ft above the water so 6.24/.1 = 62.4 ft/sec
6. derivative of 64 − 16t^2 = 32t = 64 ft/sec

We see in this post that Bill investigated all six conjectures at the same time. He started by stating his agreement and went through all the answers one by one. He showed the limiting process in Questions 2-5. The key was that he illustarted where his answers came from using the mathematics.

There was only one case where a student investigated a conjecture with mathematics and disagreed with it. This type of conjecture was useful to the mathematical discourse because the student who investigated the conjecture and proved it was incorrect mathematically. It helped ensure that correct mathematics was being practiced.

Face-to-Face Student Mathematical Discourse

Listening To, Responding To, and Questioning One Another

Listening and Responding to One Another

As in the online environment, the face-to-face students listened and responded to each other well. Nearly every idea posed from each student to the group was considered. Nearly every question was answered. An example that illustrated the usual responses that were given in their mathematical discussion came from the group's work on the diving problem from Lab 2, Question 6, Part A (see Appendix A). At the point this excerpt began, the group discovered that the derivative of the height function would lead them to find John's velocity the instant he hits the water:

> *Jayden*: The derivative is just 32*t*, right?
> *Gary*: Yeah. Negative 32*t*.
> *Jayden*: Well, yeah.

Kevin: It is 64.
Gary: 32t [pause]. Then plug in [pause].
Jayden: But why plug in [pause]? But I don't understand why we plug in 2.
Gary: 'Cause that's when he hits the water. Because it took two seconds for him to fall. So when he hits the water, it's at two seconds.
Jayden: Okay.

We can see how students listened to and responded to each other in this excerpt. Jayden began by asking Gary if he calculated the derivative correctly. Because this conversation happened face-to-face, we can see that Gary was listening in two ways. First, he said, "Yeah." This comment means that he understood and agreed with at least part of what was said. Second, from the video, we can see that he was listening to what was being asked because he was looking back and forth at Jayden and verifying his paper while they conversed. This was typical of the face-to-face student activity and conversation. Then he agreed with the answer when he mentioned that the derivative was 32t. He also felt that his response needed a little modification. This feeling is manifested in the next statement in which he says that technically there was a negative associated with the 32t, given the original function. Jayden understood the situation and agreed. Then Kevin quietly chimed in, "It is 64." This comment indicated that he was listening to the entire conversation and figured that this would be a good time to mention that 32t was 64 when t was 2. Gary wanted to continue his own line of thinking as he was working on plugging in the 2 on his paper and did not respond to Kevin. This inattention showed that there were a few times when students make comments that were not responded to. There was no indication whether or not they agreed or disagreed with Kevin's response. Jayden then asked why they plugged in two. Gary was listening to Jayden's question because he responded right away. In the end, Jayden said, "Okay." This response meant that he listened to Gary's explanation and agreed with it.

Questioning One Another

In the face-to-face environment questions were asked in 22% of the comments made by the students. The types of questions students asked face-to-face varied even more than the questions asked online. Table 15 sets forth the types of questions asked with an example of each. Figure 12 shows the frequency of the types of questions asked.

As we can see, the types of questions asked in the online environment were quite diverse. However, most of them did not occur very often. Therefore, the first four major types of questions will be discussed. Again, as with the case of the online environment, lab questions will be discussed in the tools section. The last six types of questions did not affect the discussions greatly and therefore will not be discussed.

Specific mathematical questions for the group. The most frequent type of question posed to the group in the face-to-face environment asked about the specific mathematical procedures they used. A good example that is typical of this type of question was from the group's work on the second derivative in Question 1 on Lab 3 (see Appendix B). Jayden starts off the exercise for the group by asking this question, "So for the second one, we have to do the product rule with *x*, right?" There were many times when students asked the group for feedback on specific mathematical ideas they decided to employ for a particular lab question.

Questions for individuals. The next most frequent type of question asked in mathematical discussions face-to-face were questions made specifically from individuals to individuals. They consisted of the what, how, and whys of the group's work on the mathematics. These types of questions usually followed a pattern in this environment.

Table 15

Types of Questions Students Asked in Their Face-to-Face Discussions

Question Type	Example Question
Specific Mathematical Question for Group	"Do we use limits here in this step?"
How, What, or Why Question for Specific Person	"What did you get?" "How did you get that?" "Why did you do this here?"
General Mathematical Question for Group	"What do you all think about…?"
Proposition for Group Strategy	"Do we need to take the derivative and solve?"
Lab Questions	(See Appendixes A, B, C)
Did Not Hear or Understand	"What?" or "Huh?
Problem Clarification	"What's it asking for here?"
Not Related to Conversation	"Are you taking economics?"
Course Requirements	"When is this lab due?"
Group Status or Opinion	"What are we thinking about this problem?"
Understanding Question	"Do you understand?"

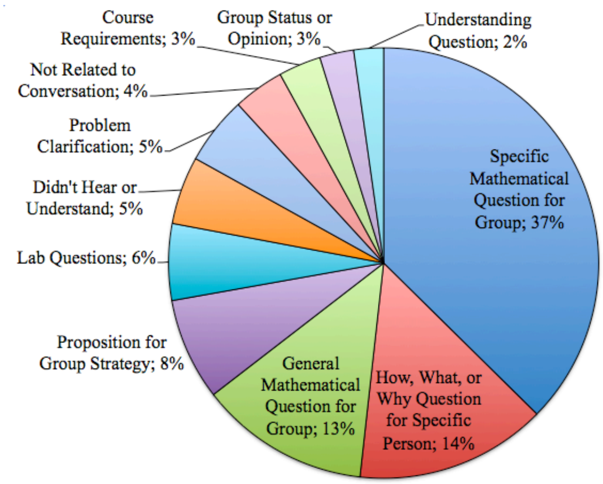


Figure 12. The various ways and frequency students asked questions in their face-to-face discussions[8].

First, when the students were in the initial stages of answering a lab question, one student would ask a second student a what question as in, "What did you get?" The second student would respond by giving their numerical answer. Then the first student would then ask a how question as in, "How did you get that?" The second student would proceed to explain the mathematical line of thinking they used to arrive at their answer. Finally, the second student would then ask a why question about a procedure used by the first student. This question prompted the first student to go into more detail about why they used that particular procedure or idea. An example that illustrates how this process usually played out in the face-to-face discussions is from the group's work on the beach problem from Lab 2, Part B, Question 3 (see Appendix A). At this point in the problem, the group was discussing the average rate people are arriving over the 15 minute interval from 3:45 p.m. to 4:00 p.m.:

Gary: What did you get for 3:45 to 4:00?

[8] Percentages were calculated using the number of the certain question type divided by the total number of questions and not the total number of comments.

⋮
Kevin: 3258.
⋮
Gary: How did you do that?
Kevin: I did $f(10)$ minus $f(9.75)$ over point 25.
Gary: So you did $f(10)$,
Kevin: Minus $f(9.75)$,
Gary: Minus $f(9.75)$,
Kevin: Divide all of that by point 25.
Gary: Why point 25?
⋮
Kevin: It's just like $f(b)$ minus $f(a)$ all over b minus a.
Gary: Okay.
Kevin: So, 10 minus 9.75.
Gary: Alright.

In this excerpt, the pattern is followed. Gary began with the what question and Kevin answered with his numerical answer. Gary followed up with the how question and asked how he got the answer. Kevin explained how he got the answer, but only superficially. He did not go into detail about where the 10, the 9.75, and the 0.25 came from or why he was using the operations he was on those numbers. However, as Kevin described to Gary his thinking, Gary repeated the procedure as he wrote it down on his paper, which was apparently enough for him up to this point. The last operation of dividing everything by 0.25 prompted him to ask the why question. Kevin's response is okay, but again he only tells Gary about the procedure and not about further background behind the procedure. Notice that the what question only drew a numerical answer from Kevin. The how question elicited a mathematical description. The why question also elicited a mathematical description. Other why questions from the data drew out the most descriptive answers.

General mathematical questions for the entire group. The next most frequent type of question that was asked consisted of mathematical questions posed to the group which were not about any specific mathematical operation, just general math topics. Two examples include, "So then what is the answer?" or "Over which interval are we looking at?" These questions brought out another level of group conversation. As in the online environment, students in the face-to-face environment discussed each other's ideas for solving the lab problems. They discussed the various specific mathematical procedures needed to find the solutions. However, in the face-to-face environment, because the conversation is fast and dynamic, there were occasions where the students needed to step back at times and remind themselves of the general mathematical problem or goal. The first example in this section showed that their conversation of mathematical operations needed to be headed toward a defendable solution. The second example in this section showed that they needed to focus on making sure that everyone was discussing the average rate over the same interval. Another group of questions similar to this one, and perhaps on the same level, were the questions labeled as "Group Status." The type of leveled conversation, described here as general versus specific questions, is unique to the face-to-face environment.

Proposition for group strategy. Another type of question that resulted to be unique to this environment was when students discussed their initial approaches to a problem. These group plans did not happen in the online environment. Such initial work was done individually. This example is from the group's work on the diving problem in Lab 2, Part A, Question 6 (see Appendix A). Gary started to find the instantaneous rate of change using the procedure which they used to find average rates of change in the previous questions. Josh also began by doing the same thing. They both found that the process gave an undefined answer:

Gary: Yes, it does [not work], which I have already [found out.] Okay, so we're just taking the derivative. Can we do that?
Josh: Sure.

They decided as a group that taking the derivative and evaluating it at two seconds would have them arrive at the correct answer. Then they adopted the plan, carried it out, and found the instantaneous rate they were looking for.

Using a Variety of Tools

Students discussing mathematics in the face-to-face environment used tools to facilitate discourse in only 37% of their comments. There were ten different types of tool use found in their discussions, which variety is comparable to the online environment. They were: (1) explanations and arguments, (2) conventional terms, (3) calculators for computations and communication, (4) invented terms, (5) graphs, (6) teacher to help with questions, (7) original lab questions, (8) hand motions, (9) tables, and (10) stories. Figure 13 shows the frequency of the types of tools discussed in the face-to-face discourse.

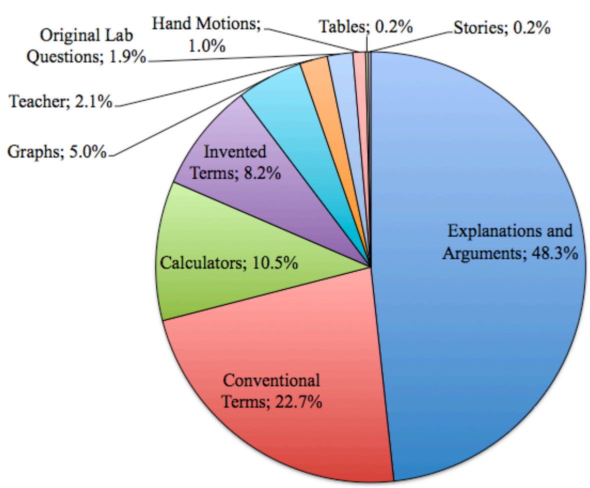


Figure 13. The various ways and frequency students used tools in their face-to-face discussions[9].

[9] Percentages were calculated using the number of comments using the certain tool type divided by the total number of comments of tool use and not the total number of comments.

Explanations and Arguments

Nearly half of the tools used to facilitate discussions face-to-face were the group's explanations and arguments. It was typical in the face-to-face environment for the students to explain the mathematics and their thinking to each other. The following example further illustrates how students explain and argue their mathematics face-to-face. This example came from the focus group's work on the beach problem from Lab 2, Part B, Question 4 (see Appendix A).

Given that the population on the beach doubled every hour, Kevin calculated that there would be 3536 people arrive at the beach in the last 36 seconds (3:59:24 p.m. to 4:00 p.m.) before the competition began. His calculation was the average rate of people arriving from the 9.99th hour to the 10th hour. He then estimated that 3540 people would arrive the instant the competition began, which was exactly at the 10th hour. Kevin's estimate was good, based on an extrapolation of the amount of people that arrived in the last minute.

Meanwhile, Jayden was working on an estimate that was based off of a closer approximation to the instant the competition began. Although Kevin's was close, it was not as exact as it could have been. Soon he stated that the estimate should be a little higher, namely 3550 people. Jayden proved that his estimate was correct when he said, "Yeah, I think it's 3550, 'cause I just did 2 to the 9.9999999 and it gave me 3548." (In his comment Jayden summarized his calculation, which was okay because it was typical of what the group was doing at the time and all understood where the numbers fit in.)

The answer 3550 was actually an extrapolation from an average rate of change where the change in time was very small. The number 3548 was from

$\frac{(5\cdot 2^{10})-(5\cdot 2^{9.9999999})}{10-9.9999999}$. This episode was given to show the precision of the mathematics that can be discussed in the face-to-face environment. The numerical difference in the two answers was small, the relative error being only around 0.3% (10/3550). However, the conceptual difference was big (3550 technically being correct, and not 3540). Jayden noticed that difference. The ability to quickly assess and work with others' responses was one of the strengths of discussing mathematics face-to-face.

Conventional Terms

When students in the face-to-face environment used terms and symbols that mathematicians use in their work, they were considered using conventional terms and symbols. When students were using tools to facilitate their discourse, they used conventional terms and symbols 23% of the time they used tools. An example of a time when Jayden used some conventional terms and symbols pretty accurately came from the group's work on the rocket problem from Question 2 of Lab 4 (see Appendix C). Jayden began by explaining that at a certain point, the velocity curve went downward until it hit zero. Kim then asked if the answer was at two seconds, which was the maximum of the velocity curve. Jayden and Ashley both said the answer was at 8.5 seconds. Kim asked why:

> *Kim*: Why would you say it's 8.5?
> *Jayden*: Because the velocity is positive until that point. Because this is the graph of the derivative, not, it's not a normal function. If it were a graph of distance over time, it would be one thing but it's not showing the distance of the rocket, it's showing the velocity.

The conventional terms that Jayden used here to describe the graph they are working with were: "derivative," "distance over time," and "velocity." The graph they were working with was the derivative of the height (distance over time) or the velocity graph. It was a key goal in this lab to understand the difference between a height graph and a velocity graph of an event. Jayden explained that the rocket was at its highest point when the velocity slows, stops for an instant, and then goes negative. If you track that on a graph of the velocity, then you need to find the point where the curve crosses the x-axis. In this case, it was near eight seconds and not two seconds. Because Jayden used conventional terms in his explanation, he communicated a lot of meaning with it. For example, when he mentioned graph of derivative, he was implying a graph where each point on it was the instantaneous slope of a corresponding point of the height of the rocket at a given time. This process comes from the benefit of using conventional terms as tools to convey meaning in mathematical discussions.

Calculators

The students in the face-to-face environment used calculators as tools to facilitate discourse about 11% of the time. There were two major ways students used the calculator in this environment. They were: (1) using the calculator as a tool to facilitate computations and (2) using the calculator as a tool to facilitate communication of mathematical ideas. Using a calculator as a computational tool, was one of the five patterns Doerr and Zanger (2000) found that students utilized when they were using calculators in the classroom. However, the article did not discuss using a calculator as a communication tool, which was evidenced in this study. Nor was the evidence of the four other uses described by Doerr and Zanger (2000).

Calculator as a computational tool. When the face-to-face students used the calculator as a computational tool, they were able to make quick calculations whose results were used in the current and subsequent problems. The following is an example of how students used their calculators for computational purposes. In this episode, the group was working on the beach problem from Lab 2, Part B, Question 1 (see Appendix A).

Josh proposed that in order to find the number of people on the beach at the time the competition began they needed to calculate $5 \cdot 2^{10}$. He did so and stated it was 5120. Gary began typing the expression on his calculator in order to verify Josh's answer.

> *Gary*: [checking with the calculator] 2 to the 10th equals [pause] So was that you're final answer? 5120?
> *Josh*: 5120, yeah, which is a ton of people.
> *Gary*: This John brings a crowd.

When he got the same answer, Josh imagined what 5120 people at a beach would be like, and Gary agreed. No one else in the data had thought about this calculation as much as Gary. It was a good way to verify the answer made sense. These types of little exchanges were frequent in the face-to-face data.

Calculator as a communication tool. The face-to-face students used the calculator to communicate mathematics when they picked up their calculator and showed their current mathematical approach to other group members. An example of how this worked in the face-to-face environment came from the group's work on the diving problem in Lab 2, Part A, Question 4 (see Appendix A).

Jayden stated that they were going to use 1.5 seconds to 2 seconds as the time interval for their calculations of the average speed during the last half-second of John's cliff dive. When Josh calculated the average rate, he said that it would be 18 2/3 feet/second if they used 1.5 in the denominator instead of 0.5. (The 0.5 in the denominator gave them a previous unrealistic answer of 120 feet/second.) After a clarification of what the current accepted answer was, Gary was still not sure about how they got the 18 2/3 feet/second. Josh decided to pass Gary his calculator in order for him to see what mathematical operations he used to get the answer:

> *Gary*: How did you get that?
> *Josh*: [passes Gary his calculator] Just from my 16 minus 1.5 squared [pause] that divided by 1.5 [Josh actually did $(64 - 16(1.5)^2) \div 1.5$ on the calculator].
> *Gary*: Why did you divide by [pause]? Oh, 'cause that's the time we are looking for. Ok, I gottcha. Thanks.

This excerpt shows that the calculator is a viable means of communicating mathematics. When the calculator was passed, Josh helped Gary see where he should start looking on his calculator screen for the information when he mentioned the 16 minus 1.5 squared divided by 1.5. Gary noticed that the answer to the first part was divided by 1.5 and did not follow that line of thinking. He asked Josh about that, but as he asked it occurred to him that 1.5 would be a difference in time as the denominator of the average rate formula. Josh's idea of passing the calculator to Gary was an effective way to show him how he came to the 18 2/3 feet/second. He was able to do this without the use of many words.

Invented Terms

About 8% of the time tools were used face-to-face, students utilized their own terms for the mathematical operations they were performing. A good example of tool use came from the group's work on the diving problem on Lab 2, Part A, Question 6 (see Appendix A). At this stage in the lab, the students were working on finding the diver's instantaneous speed when he hit the water. The face-to-face group took the approach that involved finding the derivative of the height function, which was given in the directions. They were discussing what to do with the constant term of the height function when one performs the procedure of taking the derivative. Josh stated, "The way we've been doing them, the "short-cut derivatives," [makes quote actions with his fingers] or whatever, 64 would just be nothing…" When Josh mentioned "short-cut derivatives," he was referring to using derivative rules to find the derivative of a function as opposed to applying the formal definition of the derivative to a function and finding it long-hand. Thus, the application of the term "short-cut derivatives" was invented, and was effective because the entire group understood what was intended there. This type of activity was typical of the occasions in which face-to-face students invented terms to describe mathematical processes or definitions in their conversations.

Graphs

The next most frequent tool used in the face-to-face environment was graphs. Five percent of the time students used tools, they used graphs to facilitate their discussion. Nearly every comment about using graphs came from Lab 4. The entire lab asked questions about the graph of the velocity of a model rocket from when it was launched until it hit the ground. Comments using graphs in this lab generally revolved around using the velocity graph to generate the height graph. Jayden made a comment that was typical of how the face-to-face students used graphs. The group was working on the second part to Question 11 which asked about how the graph of the height function of the rocket behaved when the velocity is positive and the acceleration is negative. Jayden stated, "So it's [the graph of the height versus time function] going to keep on going upward because the velocity is still positive, but it's just going to go at a slower rate than the first graph that we drew." The first graph in which Jayden was referring to was the corresponding height graph when the velocity and acceleration are positive, from Question 10. Here, Jayden used characteristics of the velocity graph as a tool in order to facilitate his description of the corresponding position graph.

The least frequent uses of tools in the face-to-face environment did not occur very often. Only few were the times when a student in the group asked the teacher a question. They were usually good about bringing up their concerns with each other. The students did not directly mention the lab questions very much as a tool to frame their conversations or anything else as the online students did. They mostly used conjectures and specific related questions to begin their conversations. Using hand motions was an interesting way to facilitate discourse. It was very effective to model with their hands the mathematical behavior of the lab situations, such as the model rocket from Lab 4. The idea students wanted to get across to the other group members was done efficiently this way. A typical example of this was if a student wanted to show their idea of the slope of the height graph based on the velocity graph. They illustrated with their hands going up and then down that there would be a maximum in the height graph when the velocity graph was zero. Tables and stories were used as tools, but only one time each. All these tools were potentially effective tools that were not tapped into by the students in their discussion.

Initiating Problems and Making and Investigating Conjectures and Solutions

Initiating Problems and Making Conjectures

Students working in the face-to-face environment initiated the mathematical tasks about 23% of the time. They did so in several ways (see Figure 14). They began the mathematical discussion by: (1) reading the lab question out loud, (2) posing a question about the mathematical content of the problem, (3) proposing an idea as the answer or part of the answer to the question, (4) proposing a strategy for the group to follow to find the answer, and (5) discussing the current status of the group in completing the lab task or problem. As in the online environment, the face-to-face students initiated their problems by making conjectures.

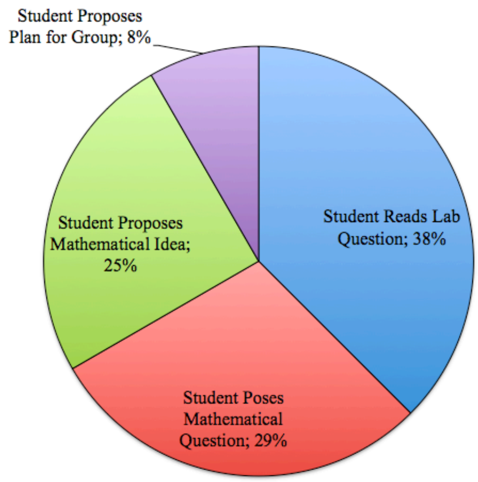


Figure 14. The various ways and frequency students initiated problems in their face-to-face discussions[10].

Student reads lab question. The most frequent way students initiated the problem tasks from the lab were by having someone read the lab question out loud. This happened 38% of the time. The example that was chosen from the data to illustrate this shows how the students' discussion evolved from working and discussing one problem to another. Here, the group went to the next question when it was read out loud by a student. This selection came from the group's work on the diving problem from Lab 2 (see Appendix A). The selection starts with the group working on Question 2 (average rate over whole dive) and then they move on to Question 3 (average rate over last second):

> *Josh*: Would it be negative 32?
> *Kim*: Ok, I thought you were doing a formula.
> *Josh*: Negative 32 feet per second? Or [pause].
> *Gary*: Except you can't have a negative speed.

[10] Percentages were calculated using the number of comments of the way they initiated new problems divided by the total number of comments from the time they initiated problems and not the total number of comments.

Kevin: Yeah, it's just speed.
Josh: Okay.
Jayden: [Reading the question] What was John's average speed during the last second before he hit the water?
Gary: 32 feet per second.
Kevin: So was that speed?
Jayden: So you need to plug in one.
Gary: From the last second.
Jayden: So it would be 64 minus 16 is that much.
Kevin: 48.
Jayden: 48, yeah. So [pause].
Gary: 48 feet per second.
Josh: Uh-huh.

The students began by discussing the final stages of the answer to Question 2. They were trying to figure out if the speed could be negative or not. They seemed to agree that it could not when Jayden read the directions to Question 3. After he read Question 3 to try and get the group going in that direction, it took a little bit before everyone was focused and discussing the new question.

At times, some students felt they needed to say a few more things about the previous question before they moved on to consider the new one. In this case, Gary was still thinking about the previous question's answer of 32 feet/second, even after Jayden read the question. Kevin asked if Gary was referring to speed or not, just before Jayden proposed a way to start finding the answer to the next question. This proposal shows that the conversations about mathematics were dynamic and not as static as the online ones.

Student poses mathematical question. The next most frequent way students initiated problems in the face-to-face environment was for a student to ask a mathematical question about their initial work on the problem. An example of this type of activity came from the group's work on the derivatives from Question 1 of Lab 3 (see

Appendix B). The excerpt begins with the group finishing up their work for the first derivative in Question 1. Then they transitioned to the second derivative in Question 1:

> *Jayden*: I got 30*x* times *x* squared plus one [pause] squared.
> *Josh*: Yes!
> *Gary*: Same! Good job team!
> *Josh*: That is the right answer.
> *Jayden*: So for the second one we have to do the product rule with *x*, right?
> *Gary*: For the first one we didn't. For the first one [pause].
> *Jayden*: Yeah. For the second one [pause].
> *Gary*: The second one we need to.
> *Calvin*: Oh yeah.

In the beginning, Jayden stated his solution to the first derivative of the first question. After which, everyone agreed by saying they got the same answer. Jayden ended this agreement phase by posing a question about the mathematics of finding the derivative of their next function. Again, everyone agreed with him. This excerpt showed that questions such as these brought the group forward in their progress more than lab questions did. Because it was a genuine question from a member of their group, they were more focused and responded to it. When the lab question was read, the students knew that it was a lab question did not respond to it as quickly.

Student poses mathematical idea. The next most frequent way students initiated their problems was for a student in the group to pose their answer, or partial answer, in the beginning of the conversation on that problem. An example of this came from the group's work on the rocket problem from Lab 4 (see Appendix C). Here is the group's discussion as they transition from Question 3 to Question 4:

> *Ashley*: I think it still is 190.
> *Jayden*: Because the velocity of where the rocket stopped burning. [pause] Yeah, it's 190.
> *Ashley*: So, it doesn't stop moving, it just stops increasing.
> *Kim*: Uh-huh, it changes direction.
> *Kevin*: So, 4 would be zero?

Jayden: Yep.

The students were working fine on Question 3, whose answer was 190 feet/second. Jayden verified the answer and then Ashley and Kim gave their agreement. Kevin brought the conversation over to Question 4 by asking the others about his answer to it. Jayden did some quick work on it and agreed with Kevin. When students posed mathematical ideas as a way to begin a problem, they were initiating a mathematical discussion. Thus, they can evaluate its validity as a group and make a little more progress in their mental and social mathematical construction.

Student proposes plan for group. The last way in which face-to-face students initiated problems occurred about 8% of the time. There were only a few times when students started their problems and someone would try and get the group to collaborate on finding the path to their answer. An example of this was when the group was working on the diving problem from Lab 2, Question 6 (see Appendix A) and transitioning into the last part of that question.

> *Josh*: Jayden and Kim, what do you guys think?
> *Jayden*: I think, yeah, 64 looks right. What do we have to write about "How do you know?"?
> *Gary*: I don't know [pause]. We know [pause].
> *Jayden*: Because the limit of that function is negative 32*t*, so at 2 seconds, when he hits the water [pause].

The excerpt began with Josh asking Jayden and Kim what their thoughts were on the answer to Question 6, the instantaneous rate in which diver John is going when he enters the water. The group's conversation transitioned into the next question when Jayden asked the group about making a plan to answer the next question. This type of question again brings the group immediately to the topic at hand because it was not just a lab question they all knew was coming, but a genuine question from a group member.

Investigating Conjectures and Solutions

As students discussed mathematics in the face-to-face environment, 60% of the time they discussed conjectures and solutions. While they were thus discussing conjectures and solutions, 62% of the time they investigated certain ones further. Then, in 53% of those group investigations, the students used mathematical evidence to support or refute the preliminary conjecture or solution. When students investigated a conjecture or solution, they: (1) agreed with the conjecture or solution without discussing the mathematical evidence, (2) gave clarifications to a conjecture or solution, (3) gave additional explanations to further illustrate conjectures and solutions, (4) agreed with a conjecture or solutions and showing mathematical evidence to support, (5) disagreed with showing mathematical evidence, and (6) disagreed without showing mathematical evidence. Figure 15 shows the frequency of the ways students investigated conjectures face-to-face.

Agreement without mathematical evidence. This type of investigation of conjectures and solutions occurred most frequently. It was usually a brief assessment of the conjecture/solution. The students would usually listen to a conjecture and then think about its correctness. The conjectures made were usually mathematically correct. Usually, all that a student wanted to do was to understand the process, agree with it, and move on. Comments made by students that portrayed this type of investigation were: “Okay,” “It sounded good to me,” “I agree,” “Right,” or “Yeah.” This type of conjecture posing and confirmation represented the majority of the face-to-face conversations.

Clarifications. The next most frequent way students in the face-to-face environment investigated conjectures was to make clarifications on proposed conjectures

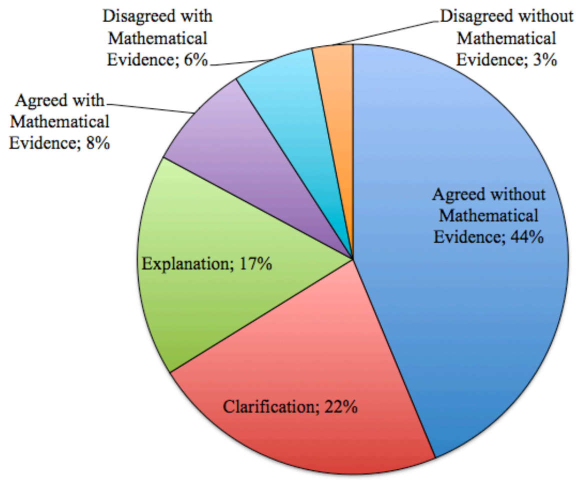


Figure 15. The various ways and frequency students initiated problems in their face-to-face discussions[11].

or solutions. There were times when certain students in the group did not understand where another student came up with their conjecture or solution. The students then investigated the conjecture further by making clarification statements or asking clarification questions. An example of this type of investigation came from the group's work on the diving problem from Lab 2, Part A, Question 4 (see Appendix A). This excerpt began with Jayden making a conjecture about the proper procedure in a specific calculation pertaining to the problem:

> *Jayden*: And then you would have to put 60 over point 5.
> *Gary*: Did you put 60 times 2.5 or did you say point 25?
> *Jayden*: Sixty over point 5. Oh, 16 times point 25.

After Jayden made his conjecture, Gary asked a clarification question. This question caused Jayden to reexamine his conjecture and investigate it further. After he did, he

[11] Percentages were calculated using the number of comments of the way they investigated conjectures and solutions divided by the total number of comments when they investigated conjectures and solutions and not the total number of comments.

realized that he had made an incorrect operation. This example shows that the face-to-face students usually investigated conjectures down to single operations.

Additional explanation. The next most frequent way students investigated conjectures face-to-face was to provide an additional explanation of the conjecture either for the benefit of themselves or another member of the group. An example that illustrates this came from the group's work on the diving problem from Lab 2, Part A, Question 4 (see Appendix A). The group had been struggling to determine the average rate which the diver travels over the last half-second of the dive. They discussed several intervals to use for the average time. Kim made a conjecture that they needed to focus on average speed and reexamine the computations that went into it. Jayden gave the following additional explanation to show what they were not getting right and what the average rate time interval should be. "So, it's going to be between 1.5 and 2. We were doing it between zero and 1.5. We need to be doing it between 1.5 and 2 cause that's the last half-second." The students were previously working with the incorrect interval and thus obtaining results that did not make sense (i.e., values that were obviously too fast or too slow). Jayden's additional explanation which investigated the conjecture to reexamine their computations called the group's attention to the error and its solution.

Agreement with mathematical evidence. This activity only happened 8% of the time the students were investigating conjectures. It is one of the goals of mathematical discussions to back up your arguments with mathematical evidence. It was good that the students were able to do this some of the time. Students would make comments saying, "Yes, it is correct because…" and then explain why in response to a numerical answer. These types of comments were good illustrations of the literature on how students should talk about mathematics in mathematics classrooms (e.g., see Ball, 1993; Lampert, 1990; Yackel & Cobb, 1996).

Disagreement with mathematical evidence. This category of ways students investigated conjectures face-to-face also reflects the kind of discussions we expect in a mathematics classroom. There were times when students did not agree with a conjecture that was presented. Here is an example where Kim, Jayden, and Ashley were discussing Question 2 of the rocket problem on Lab 4 (see Appendix C). Kim was asking about the moment when the rocket reaches its highest point during the flight:

> *Kim*: Ok, so is that 2 seconds?
> *Jayden and Ashley*: Eight point five.
> *Ashley*: Wait, that's for [Question] 2, right?
> *Jayden*: Yeah.
> *Kim*: Why would you say it's 8.5?
> *Jayden*: Because the velocity is positive until that point. Because this is the graph of the derivative, not, it's not a normal function. If it were a graph of distance over time, it would be one thing but it's not showing the distance of the rocket, it's showing the velocity.

Jayden and Ashley disagreed with Kim's conjecture as they discussed it. Then Ashley made sure that Kim was on the same question that they were on. Kim asked why the rocket reached its highest point 8.5 seconds into the flight. Jayden answered that the rocket's velocity was positive until, at 8.5 seconds, the curve dipped below the *x*-axis and became negative. He further explained that this is the point they were looking for based on the type of graph they were using. It is one that showed the velocity of the rocket, and not the height, over the whole flight. It was not a usual graph like those that they had worked with in their past mathematical experiences.

Disagree without mathematical evidence. The final way in which students in the face-to-face environment investigated conjectures was to disagree with the conjecture, and not state the mathematical evidence as to why their alternative answer was the one to use. In the example that is used to illustrate this idea, the group is working on the beach problem from Lab 2, Part B, Question 3. They decided to find the average rate people arrived on the beach from 3:45 to 4:00 p.m. We look at how Kevin disagreed with the conjecture and attempted to justify his answer:

> *Gary*: What did you get for 3:45 to 4:00?
> *Kim*: The number?
> *Gary*: Yeah.
> *Kim*: I'm just doing that right now.
> *Gary*: Oh, you're doing it right now? Okay.
> *Kim*: 7.75.
> *Josh*: Yeah, wouldn't that be 14?
> *Kevin*: 3258.
> *Kim*: Okay, I did not get that right.
> *Gary*: How did you do that?
> *Kevin*: I did $f(10)$ minus $f(9.75)$ over point 25.
> *Gary*: So you did $f(10)$ [pause].
> *Kevin*: Minus $f(9.75)$ [pause].
> *Gary*: Minus $f(9.75)$ [pause].
> *Kevin*: Divide all of that by point 25.
> *Gary*: Why point 25?
> *Kim*: I'm getting a smaller number, I don't know why
> *Kevin*: It's just like $f(b)-f(a)$ all over $b-a$.
> *Gary*: Okay.
> *Kevin*: So, 10 minus 9.75.
> *Gary*: Alright.

In this selection, Gary began by asking Kim for her thoughts on the rate at which people arrived on the beach from 3:45 to 4:00 p.m. She simply stated the number 7.75. She did not say how she got it. Josh stated a number as well, 14. He also did not say how he got it. Kevin stated another number considerably higher, 3258. Gary asked Kevin how he got that number and Kevin goes into a procedural description of how he got it using operations without explanations. In the end, Gary seemed to have been successful in copying down the procedures Kevin used, but was still left with questions like what did the point 25 represent in the denominator.

Summary

With respect to mathematical discussion in the online and face-to-face environments, several aspects played out similarly, to the same degree, and several did not. Table 16 outlines the similarities and differences discussed previously.

Table 16

Overall Comparison of Online and Face-to-Face Mathematical Discourse

Discourse Aspect	Online Frequency	Face-to-Face Frequency
Listening and Responding	Nearly 100%	Nearly 100%
Questioning	29%	22%
Using a Variety of Tools	Nearly 100%	37%
Initiating Problems	34%	23%
Investigating Conjectures	69%	60%
Further Investigations	51%	62%
Mathematical Evidence	29%	53%

In both environments, students listened and responded to each other similarly, and to the same degree. Nearly all questions were answer and nearly all conjectures received feedback. Students questioned each other about the same amount in either environment, with the online environment doing it slightly more often than face-to-face. However, the types of questions posed in the face-to-face environment were more diverse. With respect to using tools in mathematical discussions, students in the online environment used them nearly all the time, while the face-to-face group did not, just over a third.

When students did use tools, each environment afforded about the same diversity, however, the types of tools chosen by the students differed greatly. The students in the face-to-face environment did not spend as much time initiating the problems in their tasks compared with the online environment. This lack of tool use is a reflection of the relative time each environment spent discussing their work compared with starting a new topic.

Students in either environment discussed and investigated conjectures and solutions for nearly the same amount of time, the online environment having a slight edge. However, I took a deeper look at what the groups were doing as they investigated conjectures. This deeper look revealed that when the students in the online environment were discussing their conjectures only about half the time did they develop them further. The face-to-face group spent about 60% of the time further developing their conjectures. In addition, as the online group was developing their conjectures, they only used mathematical evidence less than a third of the time. However, the face-to-face group used mathematical evidence more than half the time.

These results paint a picture of how students discussed mathematics in the online and face-to-face environments in this study. The next chapter will compare and contrast the discussions in the various environments in light of the categories from the framework.

CHAPTER 6: DISCUSSION

In this chapter, I discuss the similarities and differences in student mathematical discourse with respect to the aspects of listening and responding, questioning, using a variety of tools, initiating problems, and making and investigating conjectures and solutions. Along the way, I will make connections to the literature when appropriate. This discussion will serve to answer the last part of my research question. In this way, I am preparing the foundation with which I can draw conclusions and implications about this study.

Listening and Responding

The data from online environment consisted of about 430 total posts, or turns in conversation. Because of the nature of the environment, I found that many of the turns in conversation were longer and more focused than those face-to-face. Students generally said more in one turn. Thus, the listening and responding happened statically. The data from the face-to-face environment consisted of about 1400 turns in conversation. Because of the environment, the turns were short and quick, usually not as focused, but still productive as a whole. Even though percentages were used to equalize the results with respect to frequency, the difference in ability to interact, or listen and respond, will be significant as we compare the two media here and throughout the chapter (Ellis, 2001; Meyer, 2003; Pérez-Prado & Thirunarayanan, 2002; Tiene, 2000).

These affordances and constraints did not seem to affect the listening and responding as a whole. Nearly every question was answered and nearly every conjecture received feedback. Thus, we can report that students were able to listen and respond to each other nearly 100% of the time. There were a few occasions, however, where because of the environment, students were not able to or chose not to listen and respond[12].

In the online environment, there were a few times when students did not respond to a post. The students who did this were the same students who consistently did not post until the end for every task. For example, there were times when a student may have posed a question or conjecture to the group early on in the assignment and received no response from anyone in the group until the due date itself. I attribute this phenomenon to a propensity for online students to procrastinate their work until the end. This impact on discourse is a result of the environment because they have the luxury to post work at anytime.

Additionally, in the face-to-face environment, there were a few times when ideas were ignored or interrupted, thus preventing them from being taken up and discussed by the group. This incident is a consequence of the environment. For example, when multiple students were listening and responding, the group did not take up their idea because they usually discussed and molded one idea at a time in their conversations. In sum, however, I found that these possible disadvantages were not the norm and so with regard to the aspect of listening and responding, the environment did not have a big impact.

Questioning

Students asked each other questions nearly the same amount of time in both environments, with the online environment having a 7% edge. However, there were differences in how these questions played out. The entire online data set consisted of only

[12] Even though this study did not see much difficulty with the disadvantages of discussing mathematics online or face-to-face, other studies have had enough difficulty as to make it significant (e.g., Dutton et al., 2001).

five types of questions, and the most frequent question used in their posts was a restatement or copy of the lab questions that were already given to them. They did this 41% of the time they asked questions. Face-to-face students recycled the lab questions only 6% of the time. This finding shows that as students discussed mathematics in the online environment they needed to include the lab question in their response much more so than the face-to-face environment. This need for inclusion is a byproduct of the nature of the environment because the students are not together and that the conversation usually takes place over a period of days. Therefore, as they posted they needed to remind themselves and others what part of the lab they were working on. On the other hand, due to the fact that the face-to-face environment is live and dynamic, then the students do not need as much reminding of what question they are working on because the questions were discussed linearly from beginning to end and everyone usually knew what the group was conversing about.

For online students, they asked general mathematical questions for the group 24% of the time and specific mathematical questions 20% of the time. If we compare this to how often students asked those types of questions in the face-to-face environment we see that they asked general mathematical questions 13% of the time and specific mathematical questions 37% of the time. These percentages show that the students in the face-to-face environment asked more questions on specific mathematical topics than the online environment. Therefore, because students were not together discussing their mathematical ideas step-by-step at the same time, they were not able to ask as about specific mathematical ideas nearly as much. When students were apart and had to use a static environment for communication, they tended to post all their steps at once and subsequently had to ask more general questions about their work.

The last two types of questions asked in the online environment featured students asking each other about problem objectives (10%) and requirements for the course (5%). With respect to the face-to-face environment, students asked questions about the problem objectives 5% of the time and course requirements 3% of the time. When I investigated the subsequent discourse, I found out that it did not impact the conversation and mathematical thinking much, if any.

The students in the face-to-face environment asked each other a more diverse set of questions that the students in the online environment, double in fact. The largest category of questions that students asked face-to-face, which they did not online, was questions about mathematical procedures directed to specific people (14%). Because the students were together they were able to ask questions to specific people and have a mathematical discussion with them. These questions were not found in the online data because the students were not together at the same time and therefore did not tend to pose questions specifically to other users, and thus did not have person-to-person mathematical discussions.

Eight percent of the time students in the face-to-face environment asked questions about the group's collective strategy for solving the task at hand. Again, because the students were discussing mathematics face-to-face every once in a while they wanted to specifically make sure that the group's conversation was headed in the right direction. This type of questioning was not found online, and as a consequence, the online conversations were not as cohesive and the face-to-face ones. However, all they would need to do to see where the group was in the solution process was reread previous posts, which was an advantage.

Five percent of the time, students asked questions like “What?” or “Huh?” which indicated that there were sometimes that they did not understand or hear a student’s comment. This type of interaction is something that would only happen in the face-to-face environment because the students were together, live. It is also advantageous for discussion because the repeated answer would help students’ explanations for the one asking the question and for the group. In the online environment, students would not need to ask each other to repeat phrases because when student posted comments, they were there for all to read at any time.

Four percent of the time students asked questions that were unrelated to the conversation. It is a temptation to go off topic when you are conversing with others live. The online students were always on task when they wrote, because each post had so much time and effort invested into it, which was something positive for that environment. Three percent of the questions were ones where students asked about the group status or opinion, and questions about whether or not another group member understood the discussion were asked 2% of the time, which did not affect the conversations much.

From this comparison, we see that the face-to-face environment had more diverse and more specific types of questions used as the mathematics was discussed. Thus, overall, the ability to discuss what you need to can be done with higher quality.

Using a Variety of Tools

In the online environment, students were able to use a variety of tools in nearly every post. This tool use was mostly due to the fact that students had to type out their posts. For example, they were forced to use conventional and invented terms and symbols when they wanted to show their mathematical line of thinking. This use of tools also means that overall, the online environment produced more mathematically procedural discussions, which is not as rich as conceptual discussions.

There were far fewer times a variety of tools were used in the face-to-face group percentagewise. They only used tools about 37% of the time. Because the face-to-face environment is dynamic, the students talked about other things between tool uses, thus comparatively not using tools as much as online students did. The face-to-face students would discuss explanations and conjectures, compliment each other, make plans to complete their work, talk about what they did or did not know, say "Okay," ask for answers to be repeated, and so on in their conversational turns. These comments are a result of the fact that the face-to-face environment is, again, more dynamic and flexible, not as static and restricted as it is online, which can be advantageous or disadvantageous.

It is true that there were other activities could be analyzed as tools in both environments, but in the end they were not easy to track. Therefore, I investigated tools as outlined in the literature, which definition contained a broad enough meaning for the scope of this project.

Most Utilized Tools

It is important to note that the most utilized tool in the online environment was conventional and invented terms and symbols. These tools were used about 70% of the time tools were used. Due to the fact that students were forced to type out their answers as a result of the type of environment they were working in, they learned to express their ideas through typing with conventional terms and symbols. However, when there were times they did not know how to express certain mathematical ideas. Then they resorted to finding and using their own invented terms and symbols. This use of the tool ended up happening as much as they used conventional terms and symbols, about 35% each. The face-to-face environment, however, used conventional and invented terms and symbols about 30% of the time they used tools.

On the other hand, the face-to-face environment's most frequent use of tools was explanations and arguments. They used this tool about half of the time they used tools, or if we redivide the occurrences over the entire conversation, in about 18% of their entire discussion. This percentage slightly higher when compared with only 12% of the time online students did so, since they discussed tools nearly every time. Therefore, the face-to-face environment had the slight advantage. This evidence shows that the online environment can come close to the face-to-face environment as a viable place to discuss and argue mathematics.

Least Utilized Tools

The tools that were not used frequently were environment specific, meaning that these tools function well in one environment, but not so much in the other. This characteristic is noteworthy because intuitively one might think that if there were tools that could have been used only in one environment, then those ones would be the ones the students would use most. Since this did not happen, we must take measures to capitalize on the opportunity to use tools that have the specific advantage in each environment.

In the online environment, the students had the ability, and used it to a small degree, of displaying their mathematical arguments in a format in which they could find and communicate mathematics quickly and easily (i.e., through Word attachments and WebEQ). If students did not use these tools, then typing mathematical expressions with just text became confusing. Therefore, we must emphasize to the students that this online tool should be used more than students tended to use it.

Some students found and discussed powerful mathematics by searching through the discussions of other groups and using the ideas found there in their own group discussions. This type of investigation was beneficial. One might even think to require students to do this and to be able to interpret the other groups' approaches to the mathematical task. In that way, a better understanding of the topic can be achieved. Discussing ideas among groups did not happen face-to-face, but should have when times got rough for the entire group. There was an inherent solidarity felt among groups because the students were working together live, which impeded them from sharing.

In the face-to-face environment, students were able to use different tools unique to their environment because they were working live and together. Tools such as using graphs, hand motions, tables, and stories were beneficial for the mathematical discussion. In particular with Lab 4, the face-to-face students referred to the graph of the model rocket's velocity by pointing to it and explaining their ideas. Thus, they were able to communicate their ideas about the height and acceleration of the rocket over time. This type of communication was easily done face-to-face. However, in the online environment, all these explanations must be typed out. It was not easy for the online students, so any references to graphs were limited and hard to conceptualize for them. The ability to use hand motions was a definite advantage. The face-to-face students were able to show and discuss slopes and directions of functions (e.g., the height, velocity, and acceleration functions of the rocket in Lab 4) with their hands. Using these gestures was not possible with the online students because of the distance and time separating them.

The face-to-face students discussed tables like graphs. They had to fill out a table showing the limit of a situation in Lab 2 and in order to communicate the mathematics they simply showed each other their table and pointed to the entries therein, which is also not easily done online. The online students were only able to use and talk about graphs when they could type them up.

There was one time when a story was used as a tool to communicate a mathematical argument, which can probably be done online, but not likely because of the amount of time it would take to type it all out. These types of tools that are specific to the environment must be elicited from the students more often, because they are very effective for discussions.

Calculator Use

The use of calculators was much more evidenced in the face-to-face environment. With the aid of video, I was able to see how often and in which role calculators were used in the face-to-face discussions. They used them as computational and communicative tools. It was not so in the online environment. It is likely that students used calculators while they worked online, but the students chose not to be explicit about their use. With their posts focused on answering the questions, the online students did not discuss much about how they took their operations and calculated their answer. Thus, the evidence is not in favor of the online environment with this tool, because it was not possible to determine its exact role from the evidence given.

In general, the most frequent tools used in either environment were similar, but the face-to-face environment used more arguments, explanations, and calculators. Based on the literature, this is what we want to see, so the face-to-face environment is superior here. The least frequent tools ended up being the ones that were specific to each environment and would not work well in the other. Therefore, the choice of environment does have an affect on the types of tools that are used in mathematical discussions and thus the mathematical meaning made with them. If one chooses to discuss mathematics face-to-face, there are more types of tools that are used. Additionally, the more important ones are used more frequently. Thus, more understanding can take place.

Initiating Problems

The environment had a large impact on how students initiated their mathematical tasks, and played out differently. The online environment had more flexible options for the students to begin their discussion (i.e., discussing one problem at a time, several at a time, or all at once), each with advantages and disadvantages to the resulting discussion, which is discussed in the next section. The students began nearly all their work with posts that were similar. They would begin by showing some mathematical procedures in a single problem, or multiple problems, and then they would ask the group if they made any mistakes. However, they were not consistent with how many problems they would start or how developed they would be. Because the way students initiated problems online varied, the environment did not allow for as comprehensible of a discussion as the face-to-face environment. When a student opened the discussion board and read the beginning of the mathematical discussion, they would not know what they would get.

One the other hand, the students in the face-to-face environment mostly began with someone reading the problem out loud for everyone. Or, they would be working on a new problem and someone would ask the group for help on the first procedure. Occasionally, students would start out by getting the group to agree on a plan of action for how to complete the problem together or talking about how their plan is going. This type of collaboration was not done online. What we can take away from this is that there is more collaboration and cohesion face-to-face than there is online, thus making meaning is better.

Additionally, the face-to-face environment only allowed for a linear approach to initiating their problems: discussing one problem at a time, in the order they were presented, and proceeding when everyone was ready, so that the discussion would be understandable to all. Although the amount of information that can be considered at once is limiting, the cohesiveness retained. As an interesting development in the online data, in the end the students in the online environment tended to discuss their labs linearly with one problem per post, in the order they were presented. This type of discussion was a direct transfer of the way they would discuss mathematics face-to-face. They must have realized that discussing mathematics as they would face-to-face was a better way to go. Therefore, overall, with regard to initiating problems, the environment made a difference.

Making and Investigating Conjectures and Solutions

Investigating conjectures is the vital component to any mathematics discussion and this aspect also played out similarly in each environment, but only on the surface. Students in the online environment investigated conjectures 69% of the time. This percentage is slightly above the amount of time students investigated conjectures in the face-to-face environment, which was 60%, an advantage to the online environment. However, certain pieces of data showing the amount of mathematics the students used while investigating conjectures indicated that there was more going on with this aspect than meets the eye. The thoroughness of these investigations differed considerably between the environments.

The 69% for the online environment and the 60% for the face-to-face environment only reflected the first investigation made by a student about a conjecture given previously. In the face-to-face environment, the students investigated 62% of the time conjectures deeper than with more than one look. This type of investigation happened only half of the time in the online environment. Then the amount of mathematical evidence used in these investigations greatly favored the face-to-face environment: 53% to 29%. This result means that even though students in both environments investigated their conjectures about the same amount of time, the face-to-face investigations were deeper and more mathematically sound.

When students discussed conjectures, groups in both environments showed agreement without mathematical evidence most of the time. This is the reason why it is vital for students to investigate why their conjecture was correct. How would they know for sure that the mathematical conjecture was valid? The next two most frequent types of online conjecture investigations were: (1) giving a new conjecture or (2) referencing back to the first one. About 20% of the time they did so with using mathematical evidence to prove that their new conjecture was correct. However, also about 20% of the time they did not show why their answer was correct. This type of answer would just interrupt the flow of the discussion because the student would not likely have an idea about where the answer came from. Few were the instances when a student would devote time to figuring out how the unjustified answer was obtained.

In the face-to-face environment, the next two ways of investigating conjectures were to use deeper explanations and clarifications of existing ideas, thus enriching the discussion points. This type of investigation is very beneficial for the students who did not understand the new concepts. With regard to showing agreement with a conjecture using mathematical evidence, the online group did this 10% of the time and the face-to-face group did this 8% of the time. This result was comparable, with the online group having the slight edge. The difference between disagreement with a conjecture using mathematical evidence was small along with the numbers, 3% to 1%, the face-to-face students having the edge. Clearly, how often students agree and disagree with mathematical evidence needs to be elevated.

So, in summary, on the surface it might not seem that the environment matters a lot with regard to the ability to investigate conjectures, the online environment having the slight edge, probably because of the ability to focus posts. However, deep down, there is an important difference. Because the online environment conversations are static, students did not investigate their conjectures as deeply as the face-to-face environment. They usually just posted an answer and left it. Therefore, the face-to-face environment is superior to the online environment for investigating mathematical conjectures, a discourse aspect that is highly important in the literature.

From our comparison, we learn that the online environment has many good qualities about it that enable it to be a viable arena for productive mathematical discussion. The face-to-face environment does as well. However, there are some key differences in the ways students used (or did not use) either media that have affected the resulting discourse for good and for ill. Out of the five aspects reviewed in this chapter, four discourse aspects favored the face-to-face environment overall and one fared similarly in both. We shall now make conclusions about these areas.

CHAPTER 7: CONCLUSIONS

The purpose of this thesis was to characterize student mathematical discourse in the online and face-to-face environments and then compare and contrast the resulting discussions across some general aspects of mathematical discourse. As I have done this, I have presented a picture of what it would be like to conduct a mathematics course in the online environment. I have shown the strengths and weaknesses of the environment through a comparison with the face-to-face environment, the main format we now use to conduct mathematics classes. What remains to be discussed are the conclusions we can make based on how the recent increase in online education is affecting the quality of mathematical conversations for students. As a result, the implications for teachers will also be explored. That is, what aspects of their role should they focus on as they attempt to foster online mathematical discourse? Finally, the limitations of this study and ideas for future research are also put forth.

The Online Environment's Impact on Society

The impact that online mathematics courses have on the education of our society stem from the differences they have with face-to-face courses. This study focused on student discourse in both environments. I have found that there are several key areas of praise and concern for discussing mathematics online, through the lens of the aspects of discourse used throughout this paper.

With respect to the ability to hold discussions online (i.e., listening and responding), this environment does fine. The students were able to hold cohesive and meaningful discussions with each other. Through posts, the students were able to consolidate their thoughts and edit their comments before they entered them into the conversation. This way of contributing to the conversation translated to focused work and less irrelevant comments compared with the face-to-face environment. Therefore, based on the data, we can conclude that this environment has the capacity to be a viable arena for mathematical discussion.

Concerning the manner in which the online environment handled questions, it indeed had the ability to facilitate questions. The students asked questions about as frequently as the face-to-face group. Once they were posted, the questions remained visible for students to answer and no repetition was necessary. (There were times that the face-to-face students had to repeat their ideas to another student because the listener happened to be trying to multitask or not pay attention.) Afterward, the comments and answers could be logically posted after the question through thoughtful threading. However, when the students discussed mathematics, the types of questions asked were not very deep or diverse. The face-to-face questions were. Additionally, even though the questions were answered, the discussion needed to be more elaborate. For example, a common response to a conjecture was, "Yep, that's fine." In this situation, it would be preferable if the students discussed why it was okay and what was it about the answer that you agreed with. At times students asked, "I think my answer is right, but I'm not sure. Any comments?" The other group members should have turned a vague question around and asked something like, "Well, if you are unsure of your answer, exactly which part are you unsure about? After which step do you think your calculations became inaccurate?"

The ratio for detailed questions to general questions online was 20% to 24%. For the face-to-face students, the ratio was 51% to 13%. It was good that the students asked for feedback, but they needed to be more detailed about their uncertainties in order to receive the best feedback, as the face-to-face students did. Therefore, we can conclude that the students needed to ask deeper and more detailed questions in this environment in order to have the same caliber of questions as the students in the face-to-face environment.

Relating to the use of a variety of tools to facilitate and enrich discourse, the students in the online environment used a good variety, and a couple were used quite often. However, the rest were not used as much as they could or should have been. They used conventional and invented terms and symbols well. Proper vocabulary was used and when the students did not know how to express an idea, they described it or invented a way to communicate their point. Their biggest need was to capitalize on the online tools that are inherent in that environment. For example, students did not take advantage of the ability to communicate mathematics using WebEQ and Equation Editor. These programs are specially made to type out mathematical expressions in text environments, so that mathematical ideas can be communicated easily. However, it might not have been easy for students to type up their mathematical expression because it required a lot of time to do so. In the end, the understanding gained by the group would be worth it. Therefore, some forewarning and encouragement from the teacher would be beneficial in this area.

Another tool that the online students could have used more was other groups' ideas. Using other groups could have assisted those who did not know how to start a problem. Additionally, it would have saved a lot of missed time for some groups. This tool could also have helped the group discussion when they were stuck in the middle of their problem. Graphs, tables, and stories were used in the face-to-face environment, and ways can and should be devised to use them as tools online, too (see NCTM, 1991). Therefore, we can conclude that there were times when certain tools were used effectively, namely the conventional and invented terms, but overall, the students needed to employ those effective tools more often.

With respect to the ways in which problems were initiated online, for Labs 2 and 3 the individual students began the group's lab work with whatever method came to mind and by Lab 4 most groups had a comprehensible system down. As mentioned previously, some students would post their work on the whole lab and let that be their only contribution. The rest of the group was left to sort out the answers and evaluate them for correctness. This act of posting all at once hindered the flow of discourse. In the face-to-face environment, the discourse flowed linearly, one problem at a time, and in order. By Lab 4, most of the online students were using threads logically. They did this by posting manageable sized pieces of work for discussion and evaluation as they began their lab. Online students have the ability to start the conversation about their lab with any question number. However, for the sake of the discourse of the final product, the students needed to be sure adequate space was provided for what needed to come before and after their comments. Therefore, we can conclude that the online environment does have the ability to initiate the problems so that the resulting discourse could flow smoothly, but students need to be shown the affordances and the constraints of the various ways they can initiate problems.

Pertaining to the way conjectures and solutions were investigated, students in the online environment were able to generate a comparable amount of conjectures and solutions with the face-to-face environment. However, once the students read through each other's conjectures, their subsequent analyses were lacking. Students in the face-to-face environment continued to discuss and analyze as many conjectures as they could in order to make sure they are explored thoroughly. Many conjectures with the online students were just dropped or not addressed after one assessment. In addition, students in the online environment did not use mathematical evidence nearly as much as the face-to-face environment as they investigated their conjectures, 29% to 53%, respectively. This use of mathematics, or lack thereof, is the item that illustrates the most important difference between the two environments. The online environment needs to maintain the same or higher level of mathematical verification in their investigations in order to minimize its impact on society. The lack of using mathematical evidence is most likely due to the fact that it was not easy for students to type up mathematical expressions in their posts and the time it took for students to type up explanations. Therefore, we can conclude that the online environment needs to use more mathematical reasoning in all their comments in order to be comparable to the face-to-face environment.

Lastly, we can reaffirm that the online environment allows the students to discuss mathematics in many ways like the face-to-face environment. However, there are a number of areas within the aspects of discourse that the students need to be careful with while discussing mathematics online, because of the differences between online environment and the face-to-face environment. If the students were prepared to take on mathematical discussions in the online environment, the resulting discourse would be far richer and more comparable, if not better, than the face-to-face environment. Here is where the teacher can help.

Implications for Teachers

With the aspects of student mathematical discourse characterized, a comparison made between online and with face-to-face environments, and areas to be improved identified, the implications for teachers now becomes more vital than ever. It is the teacher who is the one in charge of seeing that the students discuss mathematics effectively in whatever environment. Again, the aspects of discourse can be used as a lens to illustrate these implications for teachers.

Due to the fact that students usually listen and respond to each other well online, the teacher needs to ensure that students are participating throughout the conversation. For example, there were a few times when students would only post once and be done with the assignment. The teacher should encourage these students to develop the ideas they initially posted along with the other ideas from the group.

We concluded that the students needed to ask deeper and more detailed questions in the online environment. One of the best ways that students can ask better questions is to follow a model. The teacher should be the one to show the students how to question the ideas discussed, if the students do not know how. By questioning individuals about specific items and making comments to get the students to ask questions to coordinate their efforts (e.g., questions like, “Will you do Part B if I do Part A?”) the resulting discussion would be more effective.

The teacher can also motivate the students to use the tools provided for effective mathematical discussions, like WebEQ and Equation Editor, in order to produce more effective discussions. Many students did not use these tools, because they were new to them. As a result, students tried to use plain text, which caused the other group members some difficulty in reading and understanding their mathematical reasoning. If the teacher modeled the use of these tools in the beginning, the rest of the class would go better. Additionally, explicit calculator use could also be encouraged as another method for making conjectures and solutions. If the technology permitted, the teacher could also show how to insert graphs and tables into posts for the students to be able to see more mathematics instead of relying on typed out descriptions, which are sometimes hard to follow. Stories were also suggested by the literature (NCTM, 1991) as another tool to discuss student's ideas further. The use of stories was effective in the face-to-face environment and could work here, too, if encouraged.

The teacher also has power to set up the environment to produce discourse that is readable and fluid. He or she could set up the discussion boards with threads that break up the labs tasks into effective sized pieces for discussion. For example, if a teacher were to do Lab 3 again, he or she could set up the discussion board with threads for Question 1, Parts A and B, then Question 2, Parts A and B, and encourage discussion of their similarities and differences in subsequent posts in the same thread. Then for Lab 4, it might be more effective to initially set up the discussion threads with one question per thread. These are likely to be the most effective ways to structure online discourse because in the end this is how the students had set it up for themselves. In this manner, students could begin with any thread or question they wish, and when all is said and done, the teacher or anyone else can read the discussion from top to bottom and understand the students' ideas and work.

The main implication for the teacher, with respect to how students investigate conjectures online, is for them to guide the students to investigate worthwhile conjectures further and to use more mathematical evidence in their arguments. In this manner, a similar mathematical discussion to the ones in the face-to-face environment can be preserved or surpassed. This teacher guidance needs to be accomplished in the beginning in order to establish the mathematical norms for the online course throughout the entire semester (see Yackel & Cobb, 1996). It would also require the teacher to ask the students effective leading questions throughout the course in order to get them to think about posts that might not show enough mathematical evidence or where conjectures lie uncontested. Then, at some point, the students could eventually take on the responsibility themselves of effectively questioning each other.

It must be noted that many students have not previously done a mathematics class online or have little experience with the online courses. Many of the suggestions made could be accomplished before or in the first class meeting, which could happen live in a computer lab. In this manner, the teacher would be able to show the students and have them practice the kinds of things they would be doing throughout the semester in order to have successful mathematical discussions. If it is not possible to meet live, it would probably be beneficial for the teacher to prepare tutorials on how to discuss mathematics online. This approach would save time and work by taking care of most of the issues students have when working online, and generate rich mathematical discussions early and throughout the course.

In sum, the teacher's responsibility to ensure effective mathematical discussions online is as imperative, if not more so than the face-to-face environment, because of the inherent difficulties for students to discuss mathematics online.

Limitations

Every study has its limitations. The first limitation I need to discuss came from choosing to take a minimal intervention approach as the role of the teacher in classroom discourse. Simonsen and Banfield (2006) recommended the teacher withhold commenting in certain online situations, which I did. It is because, in part, they also saw that the students themselves would step up and help each other with their needs, like answering questions or clarifying mathematical topics. In an effort to make the approach to both media as equal as possible, I maintained the same minimal intervention for the face-to-face environment. I would have intervened to help the student reason, make connections, and solve problems according to the recommendations of NCTM (NCTM, 1991). For example, Rob asked me questions about his group's ideas (see "The Teacher as a Tool, pp. 68-69). I would have made a post of leading questions that would redirect his ideas back to him, and see what he thought he should do. I would have checked the conversation later to see if my questions were answered. It was apparent that Rob knew what actions to take in order to simplify, etc., but he was probably appealing to the teacher just to be sure that they were correct actions. I would have also confirmed the correct ideas, and by so doing, given him more confidence in his mathematical reasoning.

As another result of minimal intervention, the richness of the students' discourse was jeopardized. It might have ended up as richer had I intervened and prodded the students to think and evaluate more. For example some, albeit few, online groups had two or three posts for the entire lab. There were a few times when students' procrastination was another issue that affected their discussions. Because of procrastination, they did not end up finishing their work. The students would have been contacted personally and motivated to work. This action would have been taken if a student was found to not contribute to the face-to-face discussion after a day or two while the rest of the group was.

As another limitation, I would have also conducted whole class discussions of the online and face-to-face labs at the end of the week or class, respectively. This action would help the data be more consistent with the literature on discourse (see Sherin, 2002). The time taken in whole class discussions is a good opportunity for all students to synthesize their understandings of the new mathematical topic.

Future Research

There are a number of directions future research in this area can go based on this study. First, a study could be done that would include finding a way to assess the differences in learning, that goes further than a simple grades comparison between online and face-to-face courses. Discourse analysis and student interviews could be used to assess learning. Alternative questions to those asked in the majority of the literature could include: What would the knowledge structure of derivatives look like if a student takes a course face-to-face? How would it compare to the student's knowledge structure of derivatives look like if a student takes the same course online? What specific topics are underdeveloped or missing? For example, face-to-face students might tend to learn the procedures, concepts, and be able to make connections among them, but the online environment only procedures. People need to know these affordances and constraints so they can make an informed decision, based on what they would tend to gain or lose, by choosing between the two mediums.

Another possibility for future research would be a quantitative analysis of the types of comments used in the online and face-to-face discourse. The data, similar to that of this thesis, could be parsed into utterances. Each utterance could be coded and put into categories using the constant comparative method. Ideas for the categories could be derived from the findings in Simonsen and Wohlhuter (2007) and interpretations of higher-order thinking stemming from Herrington and Oliver (1999). Then one could compare which environment allows for more student mathematical discussion in higher-order levels.

The last possibility for future research that will be discussed could be an analysis of discourse around different mathematical topics. How do students discuss limits or integrals compared with the discussions of derivatives from this study? How do the aspects of online and face-to-face discourse change as you change topics? Are more complex topics discussed with the same depth online and face-to-face? Even when the teacher properly facilitates the discourse? As a result, how does the role of the teacher need to change as students discuss more complex topics? An indication that this could be a valuable study came from the way students were talking in Lab 2 compared with Lab 3. Lab 2 discourse flowed smoothly and the students were not stuck much. Lab 3 discourse was very procedural and the students were stuck more often. In addition, it would be a good study to do so teachers can anticipate how the students would discuss a mathematical task before it is given.

Final Remarks

The purpose of this study was to characterize the student discourse in an online and face-to-face format and thus find the similarities and differences of discussing mathematics in either media in order to evaluate online mathematical discussions. One of the reasons this comparison was done was to try to see how online and face-to-face mathematical discussions are similar and different. According the literature and the results of this study, it is conceivable that both formats can be used to effectively teach students mathematics if the teacher fulfills his or her role as the discourse moderator. Both environments can allow students to propose mathematical ideas and defend them with mathematical arguments. They both provide opportunities for mathematical errors to be corrected and explained. However, a teacher does have to keep in mind that students have certain tendencies that interfere with the flow and content of discourse. For example, in both environments, they cannot let individual student questions go unanswered or let students fail to justify their ideas mathematically. However, if the online environment is not used carefully, the students would miss out on some proper mathematical arguments. The role of the teacher is still ever important as they guide and ensure rich mathematical discussions, eventually promoting the most learning possible for every student. This study is just the beginning of the long journey required to understand the impact of online education on society, especially with regard to student mathematical discourse. We must move forward in the research to find and implement the best ways to use this means of delivering instruction, one that appears to be here to stay.

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APPENDIX A: Lab #2—The Instant of Impact

A. John is practicing a cliff dive into Lake Powell off of a cliff that is 64 feet above the surface of the water. From principles of physics we can find that his height, h (in feet), above the water at any time, t (in seconds), during his dive is given by the function:

$$h(t) = 64 - 16t^2$$

1. How many seconds after he started his dive did John hit the water?

2. What was John's average speed as he fell from the top of the cliff into the Lake?

3. What was John's average speed during the last second before he hit the water?

4. What was John's average speed during the last half-second before he hit the water?

5. What was John's average speed during the last tenth-of-a-second before he hit the water? (Questions 2-5 show how a limiting process can help find an instantaneous rate.)

6. What was John's speed at the instant he hit the water? How do you know?

B. When John arrived at 6:00 a.m. to start practicing his dive, he brought 5 friends with him. His friends found a convenient spot along the beach from which they could watch John's practice dives. During the day, other people arrived at the beach—some to play, and others in anticipation of the cliff diving competition to be held later that afternoon. In fact, the population of the people on the beach has been growing exponentially, doubling every hour.

1. If the only people on the beach at 6:00 a.m. were John's five friends, and the beach population has been doubling ever hour, how many people will be on the beach at 4:00 p.m. when the cliff diving competition begins?

2. On average, at what rate (in people/hour) have people been arriving at the beach during the day between 6:00 a.m. and 4:00 p.m.?

3. Use your own limiting process to find the instantaneous rate of how many people are arriving at the beach when the competition begins. Decide on five intervals that get smaller and smaller toward the time the competition starts. Calculate the rates of how many people arrived at the beach over those intervals.

Time interval (e.g. 12:00 p.m. to 4:00 p.m.)	Average rate that people arrive

Show work here:

4. At what rate are people arriving at the beach at the instant the diving competition begins? How do you know?

APPENDIX B: Lab #3—Using Derivative Rules

Sometimes derivative rules, such as the product rule, quotient rule, and chain rule are used in combination with each other.

Find derivatives of each of the following functions. For each pair, describe how they are alike, and how they are different.

1. $y = 5(x^2 + 1)^3$ $\qquad$ $y = x(x^2 + 1)^3$

2. $y = \dfrac{(x^2 + 2x + 1)^3}{3x^2 + 1}$ $\qquad$ $y = \left(\dfrac{x^2 + 2x + 1}{3x^2 + 1}\right)^3$

3. $y = \sqrt{4 + x^2}$ $\qquad$ $y = \left(\sqrt{4 + x^2}\right)^5$

4. $y = 4e^{x^2+1}$ $\qquad$ $y = xe^{x^2+1}$

5. $y = (x^2 + 2x + 1)^3$ $\qquad$ $y = \ln(x^2 + 2x + 1)^3$

APPENDIX C: Lab #4—The Velocity of a Model Rocket

When a model rocket is launched, the propellant burns a few seconds, accelerating the rocket upward. After burnout, the rocket coasts upward for a while and then begins to fall. In order to keep the rocket from breaking when it lands, a small explosive charge pops out a parachute.

The figure below shows velocity data from the flight of a model rocket. Use the data to answer the following questions. Explain how you determined the answer to each question.

1. For how many seconds did the engine burn?
   How do you know?

2. At what time did the rocket reach its highest point?
   How do you know?

3. What was the velocity of the rocket when the engine stopped burning?
   How do you know?

4. What was the velocity of the rocket when the rocket reached its highest point?
   How do you know?

5. At what time did the parachute pop out?
   How do you know?

6. Was the rocket falling or coasting upward when the parachute popped out?
   How do you know?

7. How fast was the rocket falling when it hit the ground?
   How do you know?

8. When was the acceleration of the rocket greatest?
   How do you know?

9. Find the acceleration of the rocket at 4 seconds.
   Find the acceleration of the rocket at 6 seconds.
   Find the acceleration of the rocket at 8 seconds.
   How did you find the acceleration of the rocket at each of these instances of time?

10. Over what interval of time are both the velocity and acceleration positive?
    Describe the motion of the rocket during this interval of time (e.g. Is it climbing or falling? Is it speeding up or slowing down?).

    What would the graph of the height of the rocket as a function of time look like over this interval of time? Either sketch it or describe it in words.

11. Over what interval of time is the velocity positive and the acceleration negative?
    Describe the motion of the rocket during this interval of time.

    What would the graph of the height of the rocket as a function of time look like over this interval of time? Either sketch it or describe it in words.

12. Over what interval of time are the velocity negative and the acceleration negative?
    Describe the motion of the rocket during this interval of time.

    What would the graph of the height of the rocket as a function of time look like over this interval of time? Either sketch it or describe it in words.

13. Over what interval of time are the velocity negative and the acceleration positive?
    Describe the motion of the rocket during this interval of time.

    What would the graph of the height of the rocket as a function of time look like over this interval of time? Either sketch it or describe it in words.